Section 1: Fundamentals

AE435

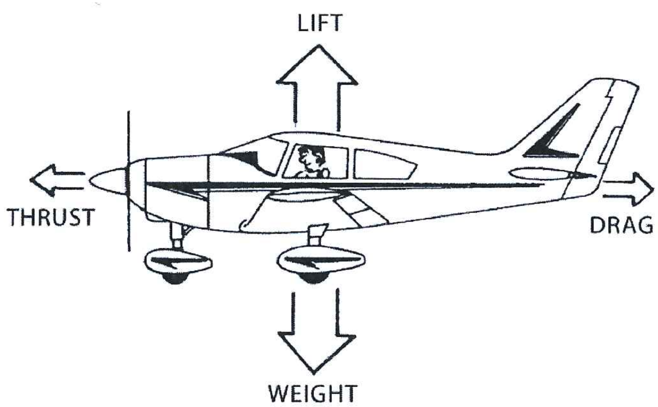
Spring 2018

1 Introduction

Contents

AE433 Aerospace Propulsion - The development of the motive force, thrust, for aerospace vehicle transportation

Consider a vehicle



$$m_{\text{vehicle}} \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = F - D_{\text{ext}} \tag{1.1}$$

Thrust - Net force developed on structure of engine due to net change in momentum in a gas or propellant stream moving through the engine.

- If F $= D_{\text{ext}}$, the vehicle is in **steady/cruise**
- If F $> D_{\text{ext}}$, the vehicle is **accelerating**
- If F $< D_{\text{ext}}$, the vehicle is **decelerating**

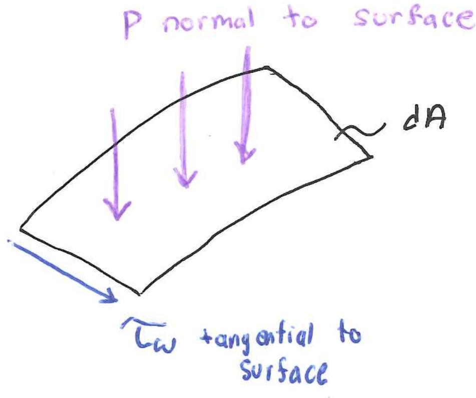
External Drag - net force developed on the external structure of the vehicle; notationly opposite in direction to thrust.

From our force balance (1.1):

- The momentum increase is opposite to the direction of the force felt by the structure.
- Momentum of gas increasing to the right, so force is to the left (Newton's 3rd law)

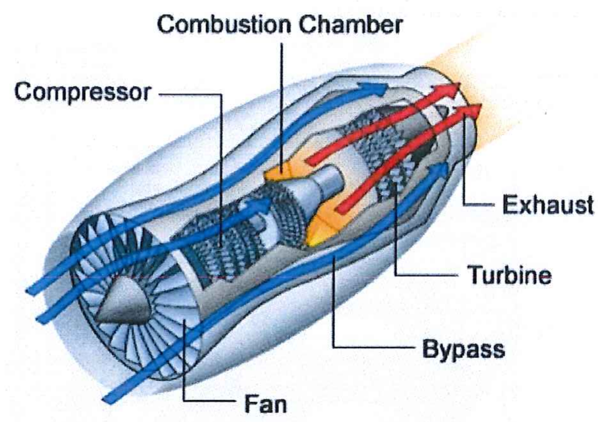
Note: The force on a solid surface of a vehicle which is adjacent to a fluid flow is the net resultant force due to...

1. Pressure $\left[\frac{N}{m^2}\right] = [Pa]$
2. Shear/Friction $\left[\frac{N}{m^2}\right]$



1.1 Classes of Propulsion Systems

1. Rockets
   - Independent of environment since they carry own propellant
   - Carries both fuel and oxidizer, all its propellant onboard
   - Good for atmospheric and space flight
2. Air-breathing
   - Use the O_2 (oxidizer) in the atmosphere and carry only the fuel. Therefore, not all the propellant is stored on the vehicle, it uses the air for oxidizer.
   - Fuels such as gas/liquid, hydrocarbon, H_2



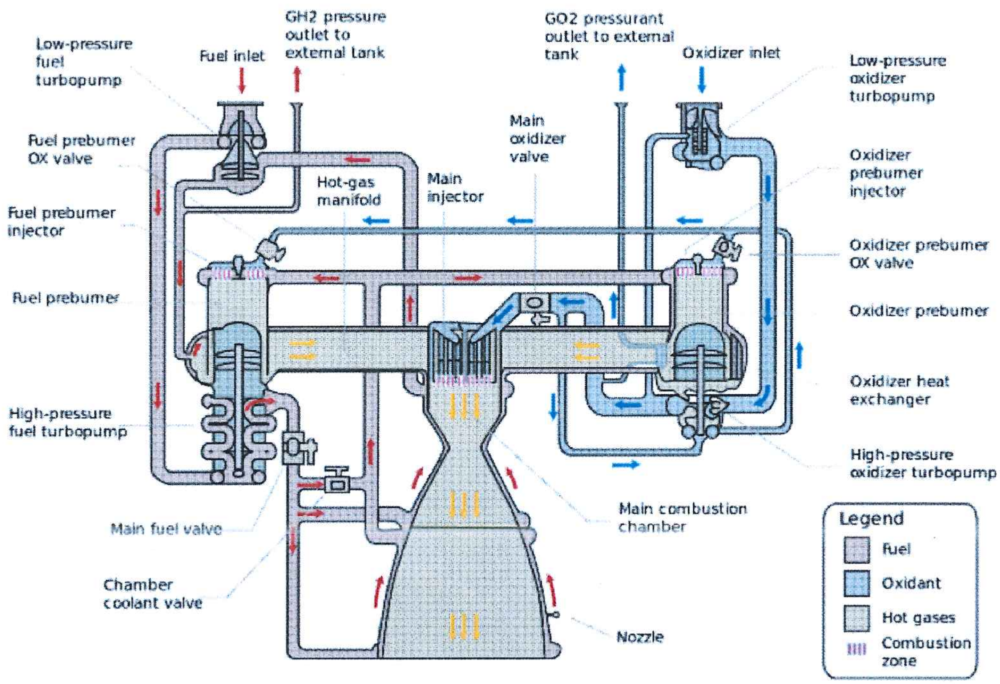
1.2 Rockets

1. Chemical Rockets

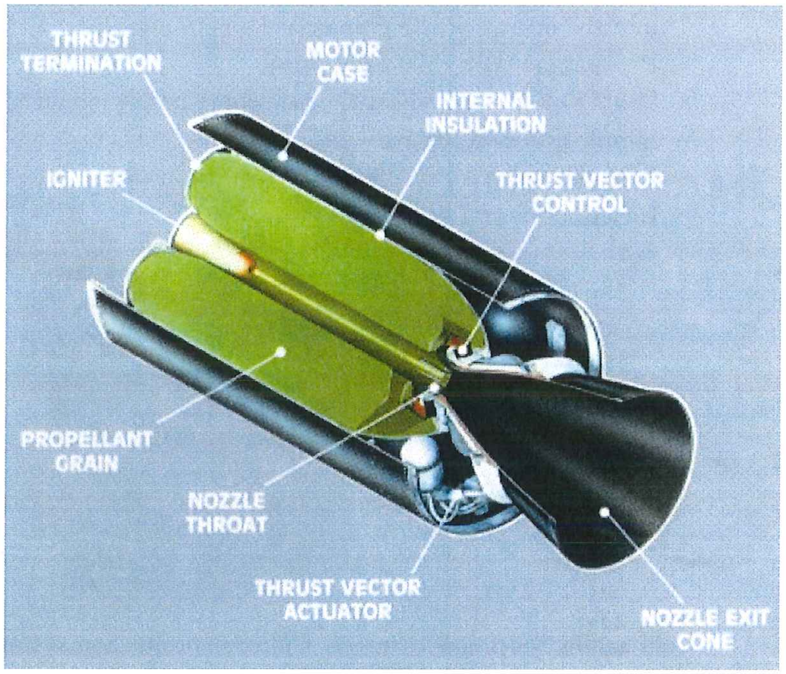
   - Energy in the propellant stream is supplies by the release of internal chemical energy via chemical reactions (just like your internal combustion engine in your car)

     (a) Liquid Propellant
       - Uses liquid propellant (fuel and oxidizer)
       - Highly energetic propellant
       - Separate fuel and oxidizer, if it is a bi-propellant.
       - Carried in low pressure, lightweight tanks
       - Will require pumps which adds weight, but allows for throttlability
       - Can shut it off

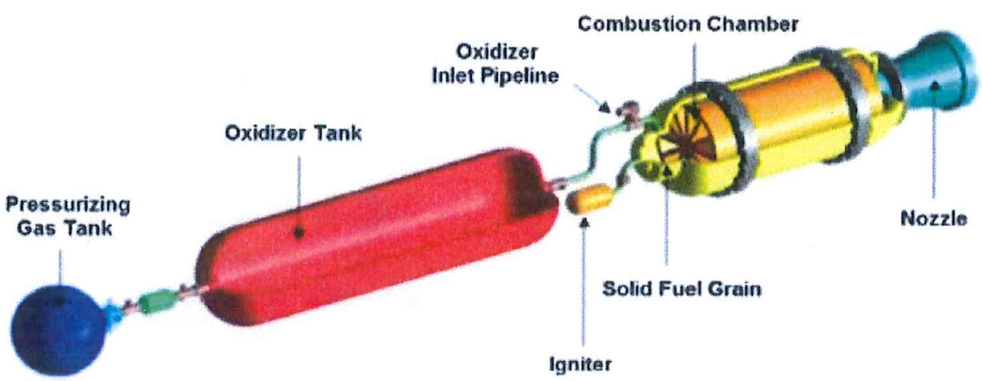
     EX: Space Shuttle Main Engine



     (b) Solid Propellant
       - Fuel and oxidizer are pre-mixed in a solid slurry which must be ignited
       - It is "simple"
       - No pumps
       - No throttle or little throttlability
       - Can use a as strap-on boosters
       - Can't shut it off
       - HTPB, PBAN, most recently: Electrical Solid Propellants



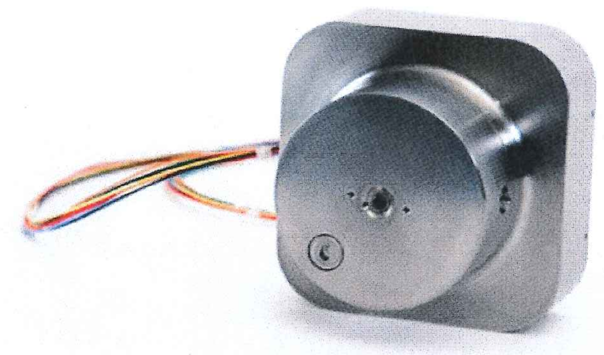
(c) Hybrid Propellant
- Fuel is solid
- Oxidizer is pumped over surface of solid fuel
- Can shut it off
- Some throttlability
- Most recently: Scaled Composites SpaceShip1, N2O nitrous oxide with HTPB



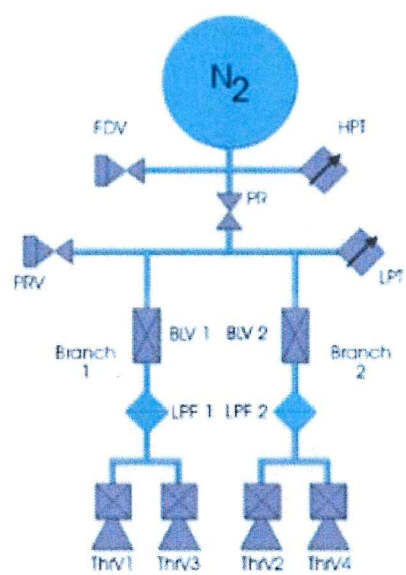
(d) Gelled Propellant
- Consistency of jelly or thick paint
- Aluminum particles can be suspended in mixture to increase propellant density
- Gel still flows with adequate pressure or shear stress

(e) Gaseous Propellant
- Cold gas jet. This is not really chemical because its not reacting with anything. The energy is not coming from the chemical bonds, its coming from the stored

internal energy of the pressurized gas in a tank.
- Pressurized tank (300-1100 psia)
- Short pulses yield adiabatic, isentropic nozzle expansion
- Simple and inexpensive
- Low efficiency
- Reaction control systems



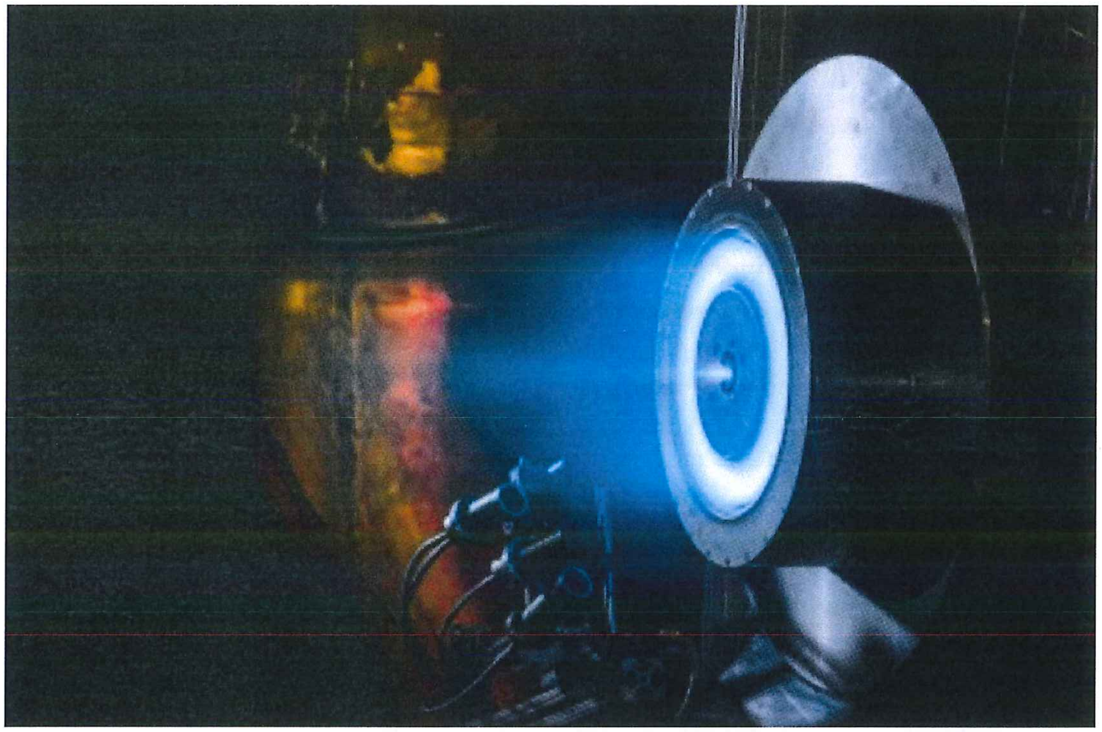
Cold gas or warm gas thruster. CubeSat propulsion system developed by CU Aerospace. they use a micro channel arc that heats up the gas. Cost: you have to put in electrical energy.



2. Non-Chemical Rockets:
   - Source of energy for propellant stream is other than chemical reaction
   - Good for very high energy missions (interplanetary) because the mass of propellant required is reduced
   - Weight of propulsion system is the key challenge

   (a) Electric Propulsion

- Electric energy used to energize propellant
- Electricity provided by solar panels, batteries, capacitors, nuclear reactions



(b) Nuclear Thermal Propulsion

- Heat from nuclear reaction used to energize propellant
- NERVA 1950s, 60s

(c) Other Propulsion

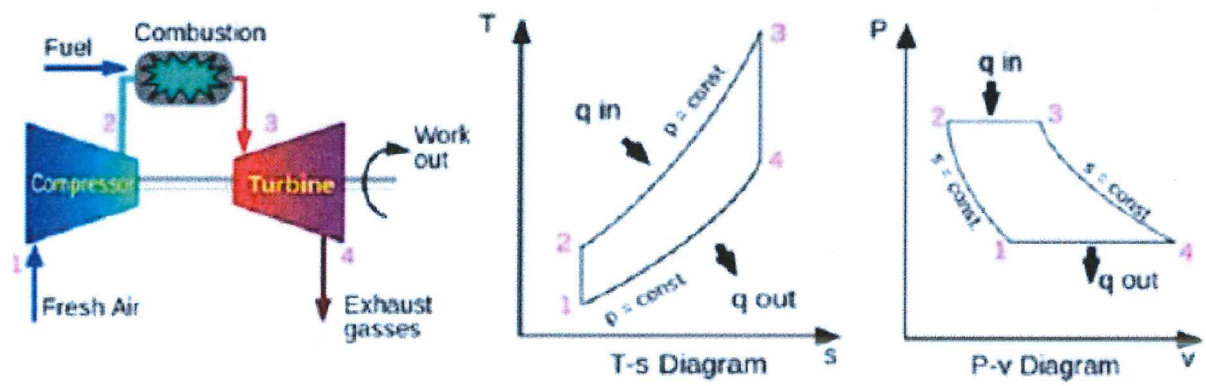
- Solar sails (solar wind)
- Tethers (jxB force, Earth B-field)
- Magnetic sail (solar wind through large current loop, jxB)
- Solar thermal

1.3 Air-Breathing

Air-breathing engines:

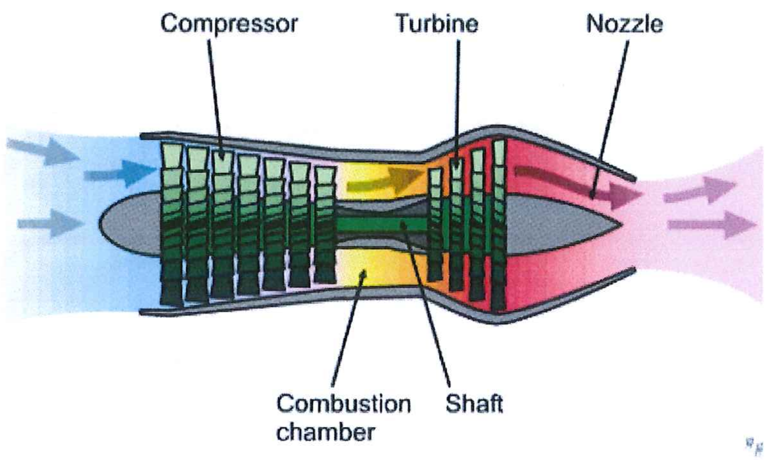
- Have inlets (intakes) to provide oxidizer for fuel. combustion
- Most Aerospace engines utilize the Brayton Cycle. The Principle is to:
  1. Induct air (slow it down) which has oxidizer
  2. Compress air mechanically to high pressure (which adds work)
  3. Add fuel and burn with oxidizer (release heat)
  4. Pass high-energy reacted mixture through turbine (extracts energy as shaft work) which runs the compressor
  5. Ejects the flow through a nozzle

Schematically:

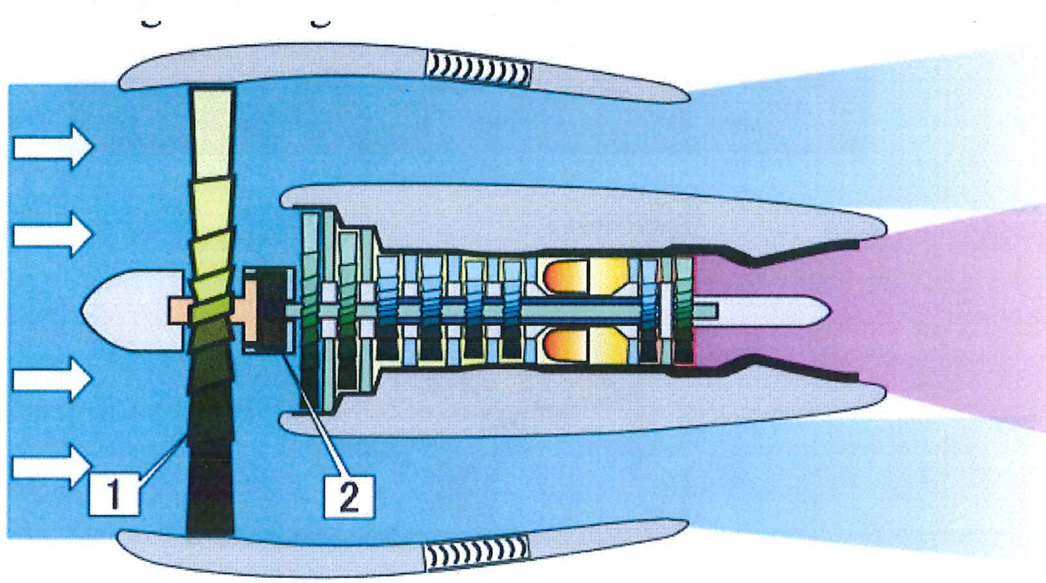


Types of AirBreathing Engines:

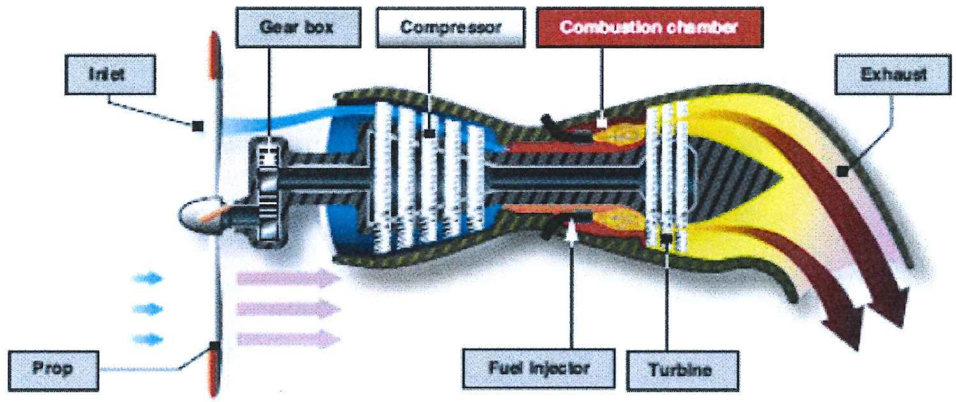
1. Turbojet
   - Basic Brayton cycle
   - Good for subsonic
   - Low supersonic (M¡2)
   - Maximum allowable turbine temperature limits Mach. Good to understand because it makes up the core of turbofan

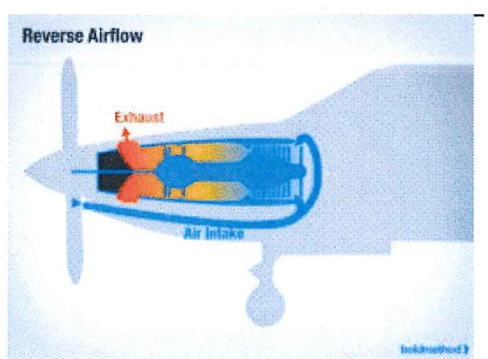


2. Turbofan (typically used nowadays)
    - Turbojet in the "core" but turbine also runs a fan on a large outer stream of inducted air
    - Good for fuel economy, but lowers speed
    - Weight drag become more of an issue

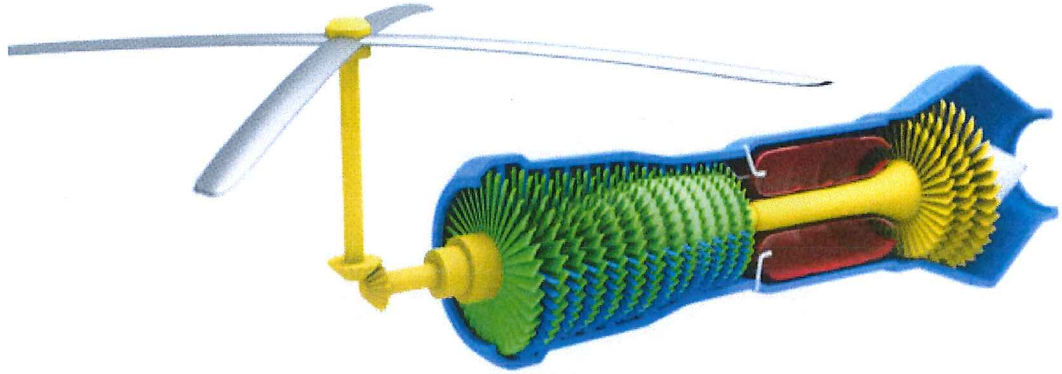


3. Turboprop
    - Like the turbojet core, but turbine also runs unducted propeller which accelerates a large mass flow
    - Losses effectiveness at speeds closer to M = 1 due to loss in propeller efficiency



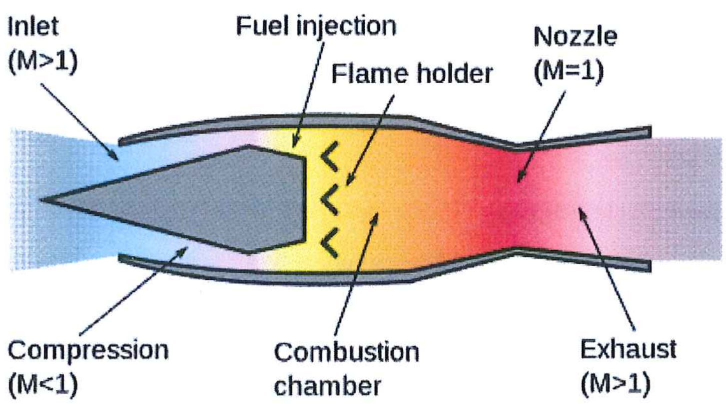


4. Turboshaft
    - Turbine work supplies to external shaft
    - Used for some helicopters, vertical take-off landing (VTOL)



5. Ramjet

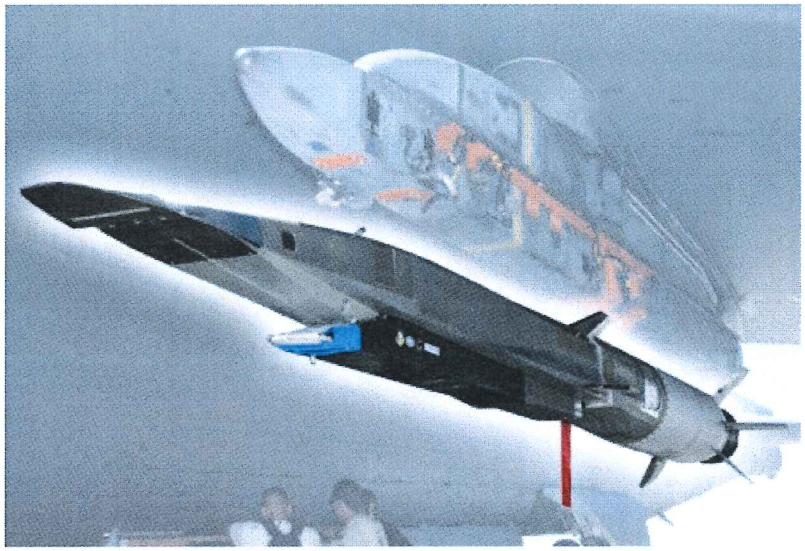
   - No turbo-machinery needed. No mechanical compressor or mechanical turbine. This engine still operates under the Brayton cycle its just all done through gas dynamic properties as oppose to machinery.
   - Relies entirely on fluid dynamic compression to get high pressures with the burner
   - Requires high flight velocity to get burn
   - Up to Mach 5 or 6



there is a strong shockwave done in front of this engine so the engine does not have to compress and we dont need turbo machinery to expand because it is all done by gas dynamic expansion.

6. Scramjet (Supersonic Combustion Ramjet)

   - X-51A Waverider (PW, Boeing, Rocketydyne, Air Force, DARPA, NASA)



    - 200 sec flight @ M=5, 5/26/2010
    - June 2011 failure, boosted to M=5, scramjet failed to start
    - Aug 2012, test crashed into Pacific, control surface failure
  - 5/2013, 1st fully successful test, M=4.8 then separates from booster, goes to M=5.1 for 240 seconds (3400 mph, 1mi/sec) on JP-7 (hydrocarbon) fuel, longest airbreathing hypersonic flight

At Mach 4 the residence time of the fluid is shorter than the time it takes to combust. IE, the air goes out of the engine before the combustion. the chemical bond energy does not have time to be released into the engine.

7. Mach up to 25 possible

8. Mixed Combined Cycle Engines

   - Uses rocket and airbreather combinations
   - E.g., SABRE engine by Reaction Engines in UK, airbreathing and rocket modes

Section 1: Fundamentals

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2 Review of Thermodynamics

Energy concepts play a major role in propulsion systems. Energy is added to a gas and then converted to thrust to propel a vehicle. We must therefore understand the science of energy, i.e., Thermodynamics

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2.1 Perfect Gas

Consider a volume, $\forall$, filled with a gas (collection of particles: atoms, molecules, ions, electrons, etc) in random motion with total mass, M.

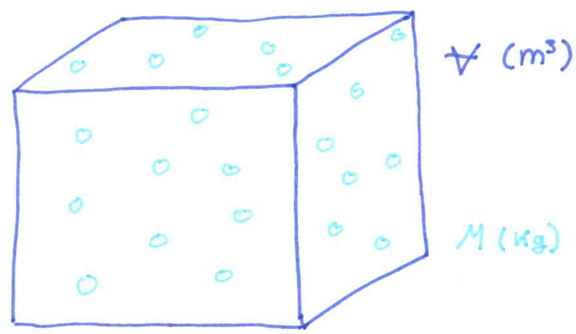


Figure: Volume with collection of particles

An observer/experimentalist could measure the properties of this system:

- P - pressure [N/m² = Pascal]
- T - temperature [K]

The relation between these properties and the mass and volume is the Equation of State.

For a "perfect gas" (and we'll talk more about what that means):

Ideal Gas Law:

$$P\forall = MRT \tag{2.1}$$

Where

- Specific gas constant:

$$R = \frac{\mathscr{R}}{MW} = \frac{\text{Universal Constant}}{\text{Molecular Weight}} \tag{2.2}$$

- Universal gas constant: $\mathscr{R} = 8314\left[\frac{J}{kmol-K}\right]$

Note that $R_{air} = 287J/kgK$.

Other forms of the perfect gas law include:

$$Pv = RT \tag{2.3}$$

Where $v =$ the specific volume $[m^3/kg]$.

$$v = \frac{\forall}{m} \tag{2.4}$$

$$P = \rho RT \tag{2.5}$$

Where $\rho =$ mass density $[kg/m^3]$.

$$\rho = \frac{M}{\forall} = \frac{1}{v} \tag{2.6}$$

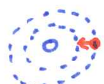
$$P\forall = N\mathscr{R}T \tag{2.7}$$

Where $N =$ number of moles of gas.

2.2 Internal Energy and Enthalpy

2.2.1 Internal Energy

Recall the microscopic picture of a gas as individual molecules bouncing around as billiard balls. These molecules have different forms of energy.
Four forms of internal energy:

1. Translational 
2. Rotational 
3. Vibrational 
4. Electronic 

[Test Thermo Ch. 0 D. gasmoleculeKE]

The internal energy of the gas is the sum of the energy contained in each of these modes for all particles in the system. We write this typically as, **E** units of [J]. In many cases, we do not word in absolute quantities, instead we work in "specific" quantities, that is, specific internal energy, **e** with units of [J/kg]. (Lower case means it's a mass specific quantity, but we usually don't say the word "mass", we usually just say "specific". So specific internal energy means internal energy per unit mass, specific enthalpy would mean enthalpy per unit mass)

If these molecules are bouncing around in maximum disorder, the system is in equilibrium.

Equilibrium - State of maximum disorder in the microscopic view

2.2.2 Enthalpy

Return now to macroscopic view:

Equilibrium (in macroscopic view) - no gradients in velocity, pressure, temperature, chemical concentrations. We have uniform properties throughout system.

Enthalpy:

$$h = e + pv \tag{2.8}$$

Enthalpy comprises a system's internal energy, which is the energy required to create the system, plus the amount of work required to make room for it by displacing its environment and establishing its volume and pressure.

This is a very important thermodynamic state variable

We typically use enthalpy because we are analyzing the fluid from a macroscopic point of view. We use a control volume approach.

Goes back to the type of systems aerospace engineers work with. 2 main types, open systems and closed systems. Aerospace usually use open-systems. Open-systems have 2 types of energy to track, internal and pv = flow work energy. The potential for a fluid to do work due the the pressure it has.

ie: pressure in a balloon, you can squeeze a balloon while holding the end and not worry because mass is constant. If you don't hold the end, the balloon is an open-system and mass is now leaving the system, the balloon is now flying around the room.

Enthalpy is a way to track internal energy and the flow work energy. The propensity for the flow to do work due to the pressure.

In general, gas can be chemically reacting, intermolecular forces important, so

$$\begin{aligned} e &= e(T, v) \\ h &= h(T, v) \end{aligned} \tag{2.9}$$

for a real Gas

2.2.3 Thermally Perfect Gas

A thermally perfect gas is not chemically reacting, intermolecular forces can be neglected, and then specific internal energy and enthalpy are only functions of one variable, e.g., Temperature.

$$\begin{aligned} e &= e(T) \\ h &= h(T) \\ de &= C_v \, dT \\ dh &= C_p \, dT \end{aligned} \tag{2.10}$$

Where

$$\begin{aligned} C_p &= C_p(T) \\ C_v &= C_v(T) \end{aligned}$$

C_v - specific heat at constant volume
C_p - specific heat at constant pressure

Specific heats for a particular fluid/gas quantify the amount of "heat" (energy) that can be stored in the fluid for a given change in temperature.

2.2.4 Calorically Perfect Gas

Now specific heats are constant.

$$
\begin{aligned}
e &= C_v T \\
h &= C_p T
\end{aligned}
\tag{2.11}
$$

Calorically perfect assumption good for most applications. Only at very large temperatures does air become chemically reacting.

"very large" temperatures, we need to be more qualitative? as scientists and engineers we MUST be quantitative.

Figure: Gas Behavior Temperature Range

2.2.5 Other Important Equations/Relations

$$C_p - C_v = R \tag{2.12}$$

$$\gamma = \frac{C_p}{C_v} \tag{2.13}$$

$$C_p = \frac{\gamma R}{\gamma - 1} \tag{2.14}$$

$$C_v = \frac{R}{\gamma - 1} \tag{2.15}$$

γ is NOT 1.4 for rocket equations.

2.3 First Law of Thermodynamics

The first law of thermodynamics relates **Heat** and **Work** interactions between a system and its environment to changes in the state of the system.

Note: Heat and Work are **INTERACTIONS**. Not a state variable. It makes no sense to say, this gas has 50J of heat, or that fluid contains 200J of work. No, we say 50J of heat was transferred to/from the system, or 200J of work was done on/by the system. Heat and work describe how systems are interacting with each other.

1. **Heat** - when two systems at different temperatures (see, different systems interacting) are brought together, an interaction occurs, this interaction is called heat (heat is transferred), δq

   Heat is transferred to or from a system.

   - Heat transfer **to** a system $\rightarrow$ positive
   - Heat transfer **from** a system $\rightarrow$ negative

2. **Work** - work is also an interaction between systems, δw

   - Work done **by** the system $\rightarrow$ negative
   - Work done **on** the system $\rightarrow$ positive

   NOTE: my sign definition for work may be different than what you previously used. What you previously used makes no sense to me. I think about heat and work based on whether they add energy to the system (a positive process) or remove energy from the system (a negative process).

Process	**Add or Remove**	**Sign**
Heat transfer TO	Adds Energy	Positive
Work done ON system	Adds Energy	Positive
Heat transfer FROM	Removes Energy	Negative
Work done BY system	Removes Energy	Negative

3. First Law of Thermodynamics

   First Law of Thermodynamics

$$\delta q + \delta w = \mathrm{de} \qquad (2.16)$$

   Note: (2.16) is only if there is no change in potential or kinetic energy of the system. (2.16) is only if heat and/or work change the internal energy of the system. Heat and work can also change the potential and kinetic energy of a system (I would hope so, how else are we going to do propulsion!? We add heat to the fluid (combustor), the fluid does work (turns the turbine, accelerates fluid to high KE exhaust)).

1st Law with Total Quantities

$$Q_{1\to 2} + W_{1\to 2} = \Delta E$$

1st Law including other forms of energy, kinetic energy and potential energy

$$Q_{1\to 2} + W_{1\to 2} = \Delta E + \Delta KE + \Delta PE$$

4. Processes

   (a) **Adiabatic** - Process with no heat transfer, $\delta q = 0$
   (b) **Reversible** - Process without dissipative phenomena, such as mass diffusion, viscosity, thermal conductivity are all absent
   (c) **Isentropic** - Process that is both adiabatic AND reversible

   Compression or expansion work is known as p dV work:

$$\delta w = -p\,\mathrm{d}v \tag{2.17}$$

Then (2.16) becomes

$$\delta q - p\,\mathrm{d}v = \mathrm{d}e \tag{2.18}$$

2.4 Entropy and 2nd Law of Thermo

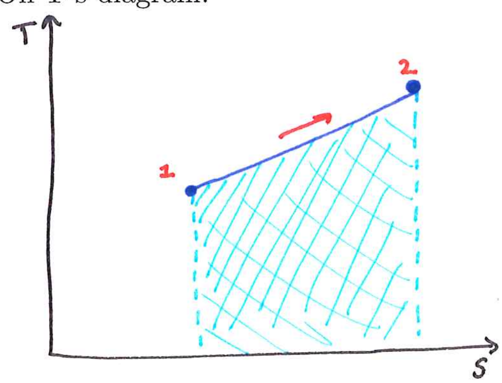
The 1st Law of Thermodynamics is an Energy Conservation Eqn. It tells us energy can be converted to different forms (internal, potential, kinetic energy), and how energy might be transferred to or from a system (via heat or work). But it does NOT tell us what direction those processes/interactions (heat , work) will take.

It is impossible for a real system to undergo a cyclic process (heat/work and return to exactly its same initial state), there is always some energy/heat loss.

The change in entropy of a system for a reversible process (one that can return to its exact initial state), is defined as:

$$\mathrm{d}s = \left.\frac{\mathrm{d}q}{T}\right|_{\text{reversible}} \tag{2.19}$$

On T-s diagram:



$$\delta q_{\text{rev}} = T\mathrm{d}s$$
$$q_{\text{rev}} = \int T\mathrm{d}s$$

Figure: T-s diagram

For an irreversible process...

Second Law of Thermodynamics

$$\mathrm{d}s \geq \frac{\delta q}{T} \tag{2.20}$$

We see that some of the energy in the system is unavailable because of entropy. Entropy represents that unavailability in system.

For adiabatic process, $\delta q = 0$ and (2.20) is then:

$$\mathrm{d}s \geq 0 \tag{2.21}$$

2.5 Calculating Entropy

Consider the 1st Law (2.18) and definition of entropy

$$\delta q - p\mathrm{d}v = \mathrm{d}e$$
$$T\mathrm{d}s - p\mathrm{d}v = \mathrm{d}e$$

$$T\,\mathrm{d}s = \mathrm{d}e + p\mathrm{d}v \tag{2.22}$$

Using enthalpy (2.8) $h = e + pv$, differentiate (2.8):

$$\mathrm{d}h = \mathrm{d}e + p\mathrm{d}v + v\mathrm{d}p \tag{2.23}$$

Combine with (2.22)

$$T\mathrm{d}s = \mathrm{d}h - v\mathrm{d}p \tag{2.24}$$

Finally, using thermally perfect gas relations (2.10), and equation of state (2.3):

$$\mathrm{d}s = \frac{\mathrm{d}h}{T} - v\,\frac{\mathrm{d}p}{T} \qquad \mathrm{d}h = C_p\,\mathrm{d}T$$
$$\mathrm{d}s = C_p\,\frac{\mathrm{d}T}{T} - R\,\frac{\mathrm{d}p}{p} \qquad \text{Equation of State, } pv = RT \tag{2.25}$$

Integrate, and assume calorically perfect, $C_p = \text{const.}$

$$S_2 - S_1 = C_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{P_2}{P_1}\right) \tag{2.26}$$

If we had used $\mathrm{d}e = C_v\,\mathrm{d}T$ instead, we would have gotten:

$$S_2 - S_1 = C_v \ln\left(\frac{T_2}{T_1}\right) + R\ln\left(\frac{v_2}{v_1}\right) \tag{2.27}$$

2.6 Isentropic Relations

An isentropic process is one that is both adiabatic and reversible, so

$$\delta q = 0 \qquad \text{and} \qquad ds = 0$$

Setting eqn. (2.26) equal to zero then:

$$\ln\left(\frac{P_2}{P_1}\right) = \frac{C_p}{R}\ln\left(\frac{T_2}{T_1}\right) \qquad \rightarrow$$

$$\frac{P_2}{P_1} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}} \tag{2.28}$$

Similarly for Eqn. (2.27)

$$\frac{v_2}{v_1} = \left(\frac{T_2}{T_1}\right)^{-\frac{1}{\gamma-1}} \tag{2.29}$$

Since $\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} \rightarrow$

$$\frac{\rho_2}{\rho_1} = \left(\frac{T_2}{T_1}\right)^{\frac{1}{\gamma-1}} \tag{2.30}$$

$$\frac{P_2}{P_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}} \tag{2.31}$$

Section 1: Fundamentals

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4 Review of Compressible Flow

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4.1 Steady 1-D Flow

Steady 1-D Flow Equations Anderson 3.3, 3.4, 3.5
Consider duct with cross section A.

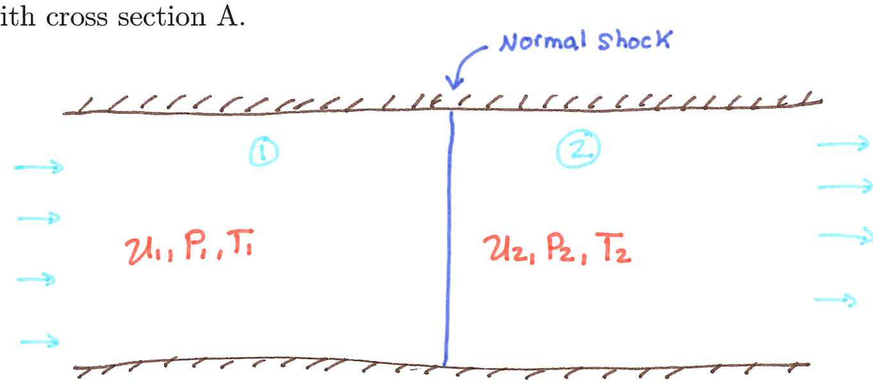


Figure: Duct with Normal Shockwave

Conservation of Mass

$$\dot{m}_1 = \rho_1 u_1 A_1 = \rho_2 u_2 A_2 = \dot{m}_2 \tag{4.1}$$

$$A_1 = A_2 = A \tag{4.2}$$

$$\rho_1 u_1 = \rho_2 u_2 \tag{4.3}$$

Conservation of Momentum

$$P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2 \tag{4.4}$$

Conservation of Energy

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2} \tag{4.5}$$

For Calorically Perfect

$$h = C_p T$$

1. Speed of Sound

   Combination of (4.3) and (4.4) for small disturbances yields speed of sound, a, as:

Speed of Sound

$$a^2 = \left(\frac{\partial P}{\partial \rho}\right)_s \tag{4.6}$$

Where subscript s is isentropic. Sounds propagation is an isentropic process.

Isentropic relations yield

$$pv^{\gamma} = \text{constant} \tag{4.7}$$

Where,

γ = Ratio of specific heat

So (4.6) becomes

$$a = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\gamma RT} \tag{4.8}$$

We define

Mach Number

$$M \equiv \frac{u}{a} \tag{4.9}$$

Where

$M < 1$ subsonic

$M = 1$ sonic

$M > 1$ supersonic

2. Total Conditions (Total state) (Stagnation Properties)

Total condition is the condition resulting from **isentropic deceleration** of the flow to zero velocity.

Conservation of Energy

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$$

Assume state 2 is the total condition

$$h_1 + \frac{u_1^2}{2} = h_o + \frac{u_o^2}{2} \tag{4.10}$$

But $u_o = 0$, so

$$h_o = h_1 + \frac{u_1^2}{2} \tag{4.11}$$

With (2.11)

$$C_P T_o = C_P T + \frac{u^2}{2} \tag{4.12}$$

Other total condition relations:

$$\frac{T_o}{T} = 1 + \frac{\gamma - 2}{2} M^2 \tag{4.13}$$

$$\frac{P_o}{P} = (1 + \frac{\gamma - 1}{2} M^2)^{\frac{\gamma}{\gamma - 1}} \tag{4.14}$$

$$\frac{\rho_o}{\rho} = (1 + \frac{\gamma - 1}{2} M^2)^{\frac{1}{\gamma-1}} \tag{4.15}$$

These Equations Tabulated in Anderson Tables, Handout 1d-Anderson Tables, Appendix A.1

3. Star * Conditions (* state)

We define a star * state as the state resulting from adiabatic acceleration (if $M < 1$) or deceleration (if $M > 1$) of the flow to $M = 1$.

Importance of this state, who cares about a "star" state? It's the missing link. We will use this state as the "link" between the flow upstream and downstream of a shock.
Define M^*

$$M* = \frac{u}{a*}$$

Energy Eqn. (4.5) with (2.11)

$$C_P T_1 + \frac{u_1^2}{2} = C_P T_2 + \frac{u_2^2}{2} \tag{4.16}$$

$$C_P = \frac{\gamma R}{\gamma - 1} \quad \rightarrow \quad \frac{\gamma R T_1}{\gamma - 1} + \frac{u_1^2}{2} = \frac{\gamma R T_2}{\gamma - 1} + \frac{u_2^2}{2} \tag{4.17}$$

with $a = \sqrt{\gamma R T}\,(eqn4.8) \rightarrow$

$$\frac{a_1^2}{\gamma - 1} + \frac{u_1^2}{2} = \frac{a_2^2}{\gamma - 1} + \frac{u_2^2}{2} \tag{4.18}$$

let state 2 be state star condition, $u_2 = a^*$ since M=1

$$\frac{a^2}{\gamma - 1} + \frac{u^2}{2} = \frac{\gamma + 1}{2\,(\gamma - 1)} a^{*2} \tag{4.20}$$

Finally, divide (4.20) by u^2

$$\frac{\frac{1}{M^2}}{\gamma - 1} + \frac{1}{2} = \frac{\gamma + 1}{2\,(\gamma - 1)} \frac{1}{M^{*2}} \tag{4.22}$$

Solve for M

$$M^2 = \frac{2}{\frac{\gamma+1}{M^{*2}} - (\gamma - 1)} \tag{4.23}$$

or solve for $M*$

$$M^{*2} = \frac{(\gamma + 1)\, M^2}{2 + (\gamma - 1)\, M^2} \tag{4.24}$$

4.2 Normal Shocks

Again consider a duct with a steady normal shock.

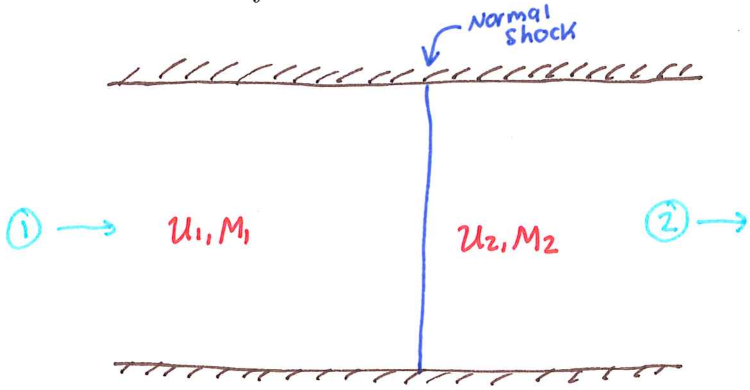


Figure: Duct with Normal Shockwave

We know state 1, how to get state 2?

4.2.1 Prandtl Relation

Governing Eqns.

Continuity

$$\rho_1 u_1 = \rho_2 u_2 \tag{4.25}$$

Momentum

$$P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2 \tag{4.26}$$

Energy

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2} \tag{4.27}$$

Perfect Gas Law

$$P = \rho R T \tag{4.28}$$

Calorically Perfect gas

$$h = C_P T \tag{4.29}$$

Combine (4.25) and (4.26)

$$\frac{P_1}{\rho_1 u_1} - \frac{P_2}{\rho_2 u_2} = u_2 - u_1 \tag{4.30}$$

Recall (4.8)

$$a = \sqrt{\frac{\gamma P}{\rho}}$$

$$\frac{a_1^2}{\gamma u_1} - \frac{a_2^2}{\gamma u_2} = u_2 - u_1 \tag{4.31}$$

Recall a form of the energy equation (4.20) (star * state), we can then write:

$$a_1^2 = \frac{\gamma+1}{2} a^{*2} - \frac{\gamma-1}{2} u_1^2$$
$$a_2^2 = \frac{\gamma+1}{2} a^{*2} - \frac{\gamma-1}{2} u_2^2 \tag{4.32}$$

a* is the same in these equations, even though one equation is written for state 1 (upstream of shock), the other equation is for state 2 (downstream of shock).

Why?

Because it's (the shock) is an adiabatic (not isentropic) process.

Plug (4.32) into (4.31) to get:

$$a^{*2} = u_1 u_2 \tag{4.33}$$

Prandtl Relation

Importance: The Prandtl Relation relates the upstream and downstream states of a shock u1 and u2 , clearly the *star state is the link between these states. From this simple relation, we will be able to relate the other state properties across a shock, e.g., pressure, temperature, density, internal energy.

4.2.2 Property Ratios

Take (4.33) → $a^{*2} = u_1 u_2$ → $1 = \frac{u_1}{a^*}\frac{u_2}{a^*}$

$$1 = M_1^* = M_2^*$$

$$\frac{1}{M_1^*} = M_2^* \tag{4.34}$$

Another form of Prandtl Relation

Now use (4.24) into (4.34), this is just another form of the energy eqn. being plugged into (4.34).

Then,

$$M_2^2 = \frac{1 + \left[\frac{\gamma-1}{2}\right] M_1^2}{\gamma M_1^2 - \frac{\gamma-1}{2}} \tag{4.35}$$

Finally, have determined downstream Mach # M_2 is only a function of upstream Mach # M_1. If we know M_1, can calculate M_2.

Now, other state properties:

Density:

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{u_1^2}{u_1\, u_2} = \frac{u_1^2}{a^{*2} = M_1^{*2}} \tag{4.36}$$

Uses 4.24 (another form of energy equation) into 4.36...

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma+1)\, M_1^2}{2+(\gamma-1)\, M_1^2} \tag{4.37}$$

Pressure: Momentum Equation yields →

$$\frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma+1}\,(M_1^2 - 1) \tag{4.38}$$

Temperature: equation of state yields →

$$\frac{T_2}{T_1} = \frac{P_2}{P_1}\frac{\rho_1}{\rho_2} = \frac{h_2}{h_1}$$
$$= \left[1 + \frac{2\gamma}{\gamma+1}\left(M_1^2 - 1\right)\right]\left[\frac{2+(\gamma-1)\, M_1^2}{(\gamma+1)\, M_1^2}\right] \tag{4.39}$$

We assumed R is constant. We assumed the specific gas constant is constant. R comes from

We assumed C_P is constant across the shock.

If we have a really strong shock wave, the temperature may get so hot that there may be decomposition so the MW of the gas is going to change so R would not be constant.

Again, these property ratios across a normal shock are tabulated in Anderson, handout 1d. Note: They are tabulated for γ= 1.4, air. If γ does not = 1.4, you must calculate!

4.2.3 Final Normal Shock Comments

What about total temperature, total pressure?

Energy Eqn:

$$C_P\, T_1 + \frac{u_1^2}{2} = C_P\, T_2 + \frac{u_2^2}{2} \tag{4.40}$$

$$C_P T_o = C_P T + \frac{u^2}{2}$$
$$C_P T_{o1} = C_P T_{o2}$$
$$T_{o1} = T_{o2} \quad \text{if} \quad C_P = \text{constant} \tag{4.41}$$

Total temperature is constant across shock.

What about total pressure?

Entropy Eqn (2.26)

$$S_{o2} - S_{o1} = C_P \ln\left(\frac{T_{o2}}{T_{o1}}\right) - R \ln\left(\frac{P_{o2}}{P_{o1}}\right) \tag{4.42}$$
$$S_{o2} = S_2 \qquad S_{o1} = S_1$$

Total Condition $\Delta s = 0$ with $T_{o2} = T_{o1}$ $\rightarrow$

$$S_2 - S_1 = -R \ln\left(\frac{P_{o2}}{P_{o1}}\right) \tag{4.43}$$

The change in total pressure across a shock depends on the change in entropy. But how does entropy change?

$\Delta S > 0$ across shock so... $\frac{P_{o2}}{P_{o1}} < 1$, P_o decreases

$$S_{o2} - S_{o1} = C_P \ln\left(\frac{T_{o2}}{T_{o1}}\right) - R \ln\left(\frac{P_{o2}}{P_{o1}}\right) \tag{4.44}$$

So,

$$\frac{P_{o2}}{P_{o1}} \quad \text{also a function of } M_1$$

Recall 2nd law of thermo,$\Delta S > 0$, so the total pressure DECREASES across the shock. The loss in total pressure is a result in entropy , due to irreversible effects occurring within the thin shock, thermal conductivity, mass diffusion, viscosity.

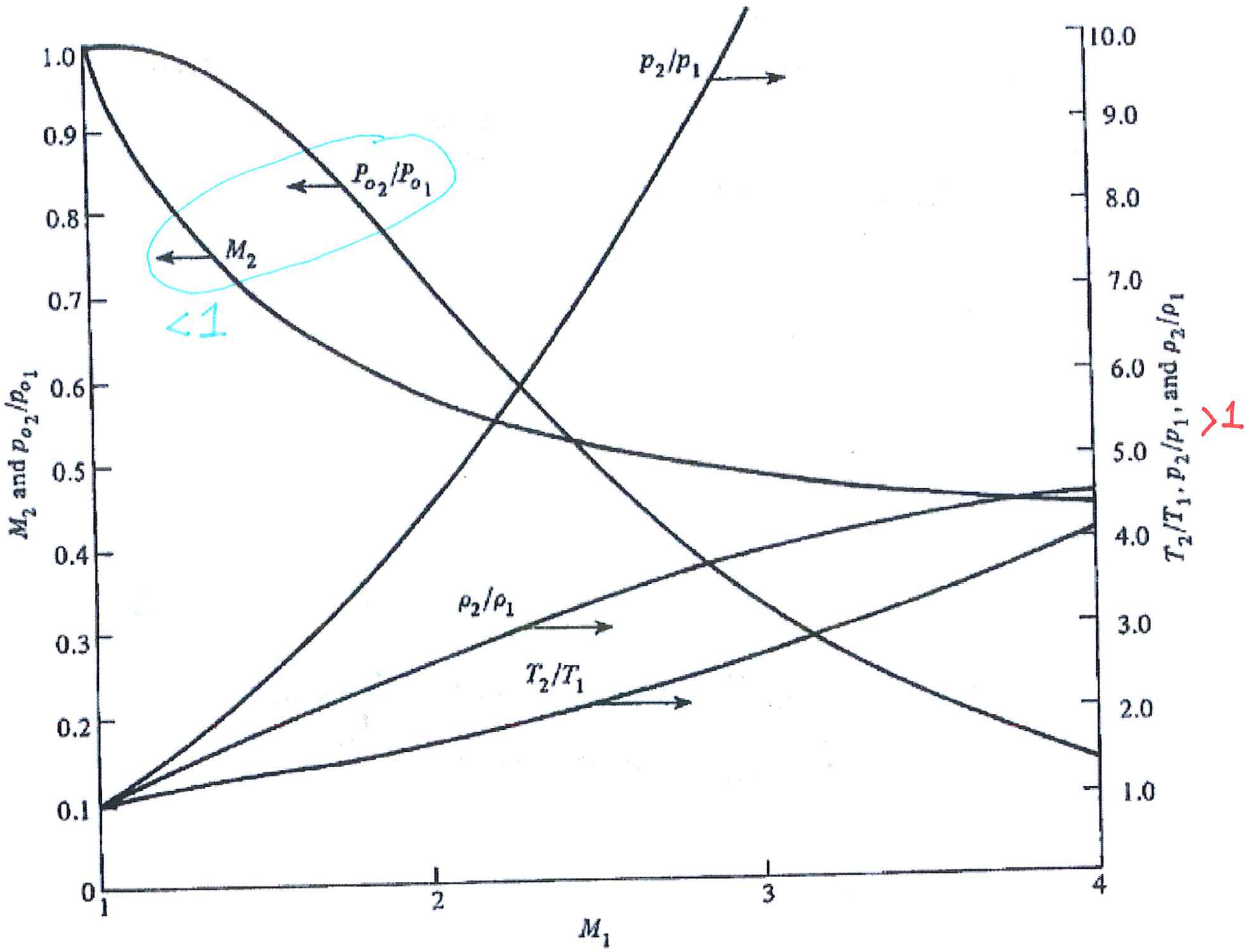


Figure 3.10 | Properties behind a normal shock wave as a function of upstream Mach number.

These equations represent a physics based model for how to predict how flow properties change across a shock wave. Like all models, there were assumptions that were made like R being constant and C_P being constant which are only good when normal shock wave is weak. We also used a calorically perfect gas model which is accurate within 4-6%. This assumptions mean it will not match up with the real world.

4.3 Oblique Shocks

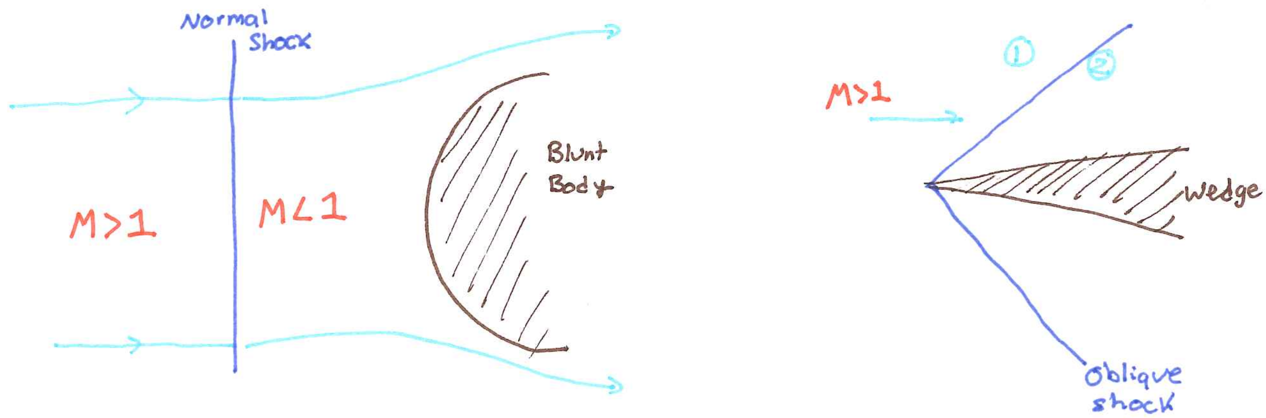


Figure: Oblique Shock Waves at the Tip of a Wedge.

How to analyze oblique shocks?
Divide incoming flow into normal and tangential components with respect to oblique shock.

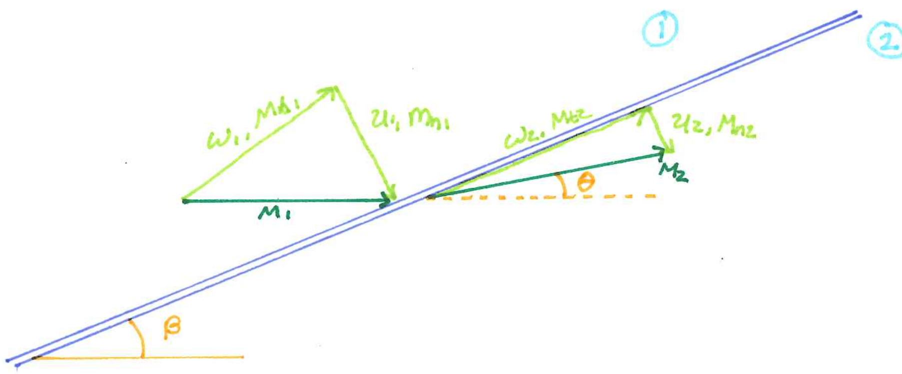


Figure: Oblique Shock Waves Components

Conservation equations show:

$$w_1 = w_2 \qquad \text{Velocity in the Tangential Component} \tag{4.46}$$

Tangential (velocity) component same across shock.

Further, normal component to the oblique shock follows the normal shock relations.

So, for oblique shock, divide the flow into normal and tangential components. Keep the tangential component the same and apply normal shock relations (section IV. B.) to the normal component.

How to get normal component? M_{n1}

From figure geometry:

$$M_{n1} = M_1 \sin\beta \tag{4.47}$$

Use (4.37, 4.38, 4.39)

$$\frac{\rho_2}{\rho_1} = \frac{\gamma + 1)\, M_{n1}^2}{2 + (\gamma - 1)\, M_{n1}^2}$$

$$\frac{P_2}{P_1} = (1 + \frac{2\,\gamma}{\gamma + 1}\,(M_{n1}^2 - 1)$$

$$\frac{T_2}{T_1} = \frac{P_2}{P_1}\,\frac{\rho_1}{\rho_2}$$

$$M_{n2}^2 = \frac{1 + \left[\frac{\gamma - 1}{2}\right] M_{n1}^2}{\gamma\, M_{n1}^2 - \frac{\gamma - 1}{2}}$$

Now need to convert back to the streamwise coordinate M_2

From geometry, figure,

$$M_2 = \frac{M_{n2}}{\sin(\beta - \theta)} \tag{4.48}$$

But need to know θ and β, angles? θ is deflection angle, for instance, β is the wave angle, how to get β?

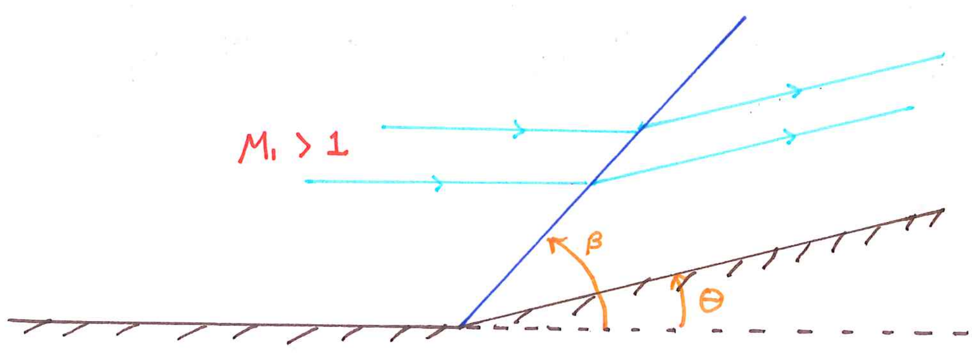


Figure: Deflection Angle

From geometry of figure above:

$$\tan(\beta) = \frac{u_1}{w_1} \tag{4.49}$$

$$\tan(\beta - \theta) = \frac{u_2}{w_2} \tag{4.50}$$

Remember that $w_2 = w_1$:

$$\frac{\tan(\beta - \theta)}{\tan(\beta)} = \frac{u_2}{u_1} \tag{4.51}$$

Combine (4.51) with (4.47), (4.37), and continuity to get

$$\frac{\tan(\beta-\theta)}{\tan(\beta)} = \frac{2+(\gamma-1)\,M_1^2\,\sin^2(\beta)}{(\gamma+1)\,M_1^2\,\sin^2(\beta)} \tag{4.52}$$

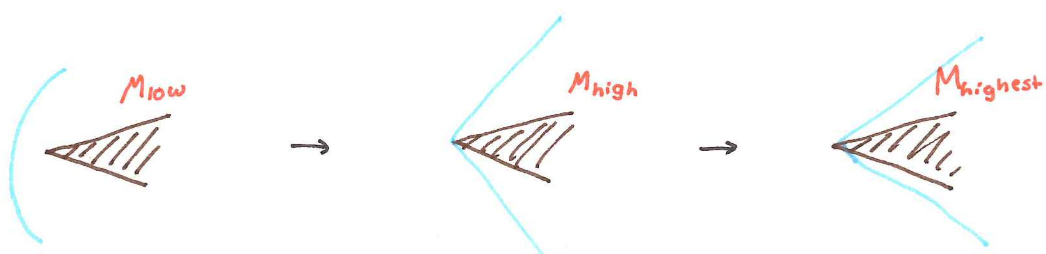
This can be rewritten as:

$$\tan\theta = 2\cot\beta\left[\frac{M_1^2\,sin^2(\beta)-1}{M_1^2\,(\gamma+\cos(2\beta))+2}\right] \tag{4.53}$$

θ - β - M relationship Handout of $\theta - \beta$ - M diagram.

From handout, notice that:

1. There is a θ_{max}, for a given M, if $\theta > \theta_{max}$, curved detached shock
2. If $\theta < \theta_{max}$, 2β solutions exist. The smaller β is the weak solution and normally occur. Only in special circumstances (for example with high back pressure) does the strong/large β solution occur.
3. If $\theta = 0$, $\beta = \frac{\pi}{2}$, normal shock, or $\beta = \mu = \sin^{-1}\frac{1}{M}$ Mach wave
4. As M increases, for fixed θ



As θ increases for fixed M

4.4 Expansion Waves

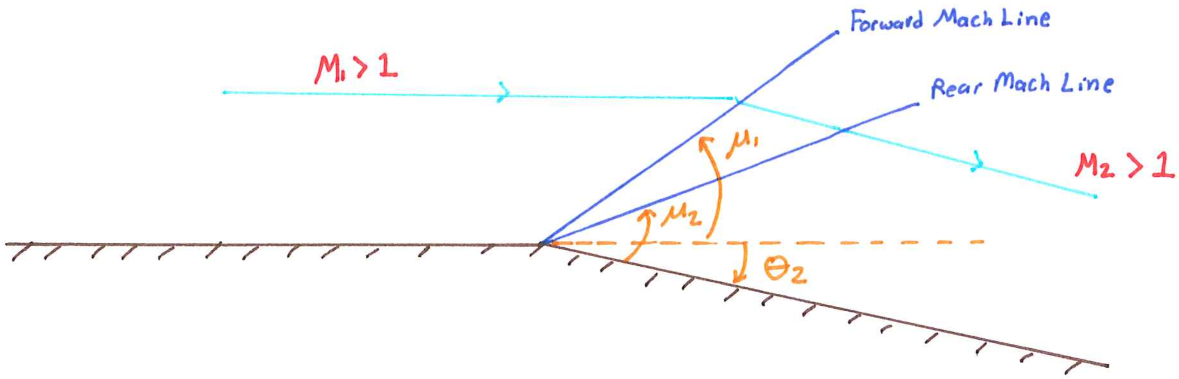


Figure: Expansion Wave

Now supersonic flow passes over expansion corner such that

1. $M_2 > M_1$
2. $\frac{P_2}{P_1} < 1$ and $\frac{T_2}{T_1} < 1$ and $\frac{\rho_2}{\rho_1} < 1$
3. Expansion is a continuous region with infinite number of Mach waves bounded by μ_1 and μ_2
4. Streamlines are smooth curves through expansion
5. Expansion is isentropic $\mathrm{d}s = 0$

If we know upstream state 1 , how to get downstream state 2?

Consider the velocity changes across a single Mach line.

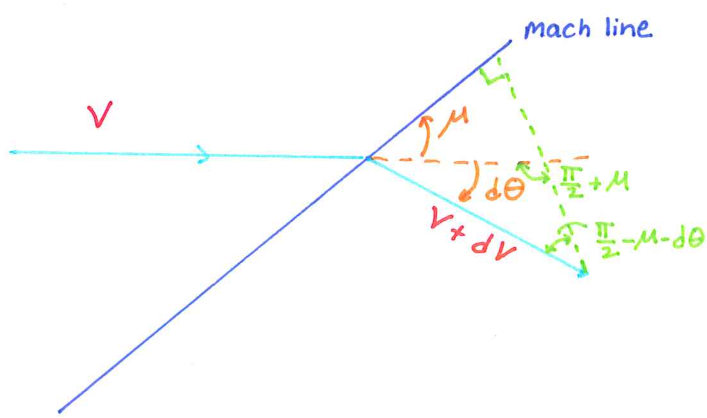


Figure: Expansion Wave Velocity Vectors

Geometry:

Law of sines

$$\frac{V + \mathrm{d}V}{V} = \frac{\sin(\frac{\pi}{2} + \mu)}{\sin(\frac{\pi}{2} - \mu - \mathrm{d}\theta)} \tag{4.54}$$

Use of trig identities

$$1 + \frac{\mathrm{d}V}{V} = \frac{\cos\mu}{\cos\mu\cos\mathrm{d}\theta - \sin\mu\sin\mathrm{d}\theta} \tag{4.55}$$

Assume small deflection angle

$$\mathrm{d}\theta \quad \rightarrow \quad \begin{aligned} &\sin\mathrm{d}\theta \sim \mathrm{d}\theta \\ &\cos\mathrm{d}\theta \sim 1 \end{aligned}$$

$$1 + \frac{\mathrm{d}V}{V} = \frac{\cos\mu}{\cos\mu - \mathrm{d}\theta\sin\mu} = \frac{1}{1 - \mathrm{d}\theta\tan\mu} \tag{4.56}$$

Taylor series expansion of

$$\frac{1}{1-x} \rightarrow 1 + \frac{\mathrm{d}V}{V} = 1 + \mathrm{d}\theta\tan\mu + \ldots \tag{4.57}$$

$$\mathrm{d}\theta = \frac{\frac{\mathrm{d}V}{V}}{\tan\mu} \tag{4.58}$$

Remember

$$\mu = \sin^{-1}\frac{1}{M}$$

So, Geometry $\Rightarrow$

$$\tan\mu = \frac{1}{\sqrt{M^2 - 1}} \tag{4.59}$$

(4.59) into (4.58)

$$\mathrm{d}\theta = \sqrt{M^2 - 1}\,\frac{\mathrm{d}V}{V} \tag{4.60}$$

Here is how a small change in angle $d\theta$ of the flow relates to the change in velocity dV of the flow.

But, we need to sum over all angles between θ_1 and θ_2 (initial and final flow angles), thus we integrate:

$$\int_{\theta_1}^{\theta_2} d\theta = \int_{M_1}^{M_2} \sqrt{M^2 - 1}\frac{dV}{V} \tag{4.61}$$

let $M = \frac{V}{a}$

$$\frac{dV}{V} = \frac{1}{1 + \frac{\gamma-1}{2}M^2}\frac{dM}{M} \tag{4.62}$$

$$\int_{\theta_1}^{\theta_2} d\theta = \theta_2 - \theta_1 = \int_{M_1}^{M_2} \frac{\sqrt{M^2-1}}{1 + \frac{\gamma-1}{2}M^2}\frac{dM}{M} \tag{4.63}$$

Prandtl-Meyer Function

$$v(M) = \int \frac{\sqrt{M^2-1}}{1 + \frac{\gamma-1}{2}M^2}\frac{dM}{M} \tag{4.64}$$

$$v(M) = \sqrt{\frac{\gamma+1}{\gamma-1}} \cdot \tan^{-1}\left(\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)}\right) - \tan^{-1}\left(\sqrt{M^2-1}\right) \tag{4.65}$$

Tabulated in Anderson tables, handout 1d.

So from (4.63):

$$\theta_2 = v(M_2) - v(M_1) \tag{4.66}$$

To answer our question, how to determine downstream state 2 from upstream state 1?

1. Calculate $v(M_1)$ using known M1 and (4.65) or table
2. Calculate $v(M_2)$ using (4.66) and known deflection angle
3. Use (4.65) or table to get M_2

Expansion waves are isentropic, so: $T_{o2} = T_{o1}$ and $P_{o2} = P_{o1}$

$$\frac{T_2}{T_1} = \frac{1 + \frac{\gamma-1}{2}M_2^2}{1 + \frac{\gamma-1}{2}M_1^2} \qquad \frac{P_2}{P_1} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}} \tag{4.67}$$

4.5 Rayleigh Flow

Rayleigh Flow - 1D Flow with heat addition

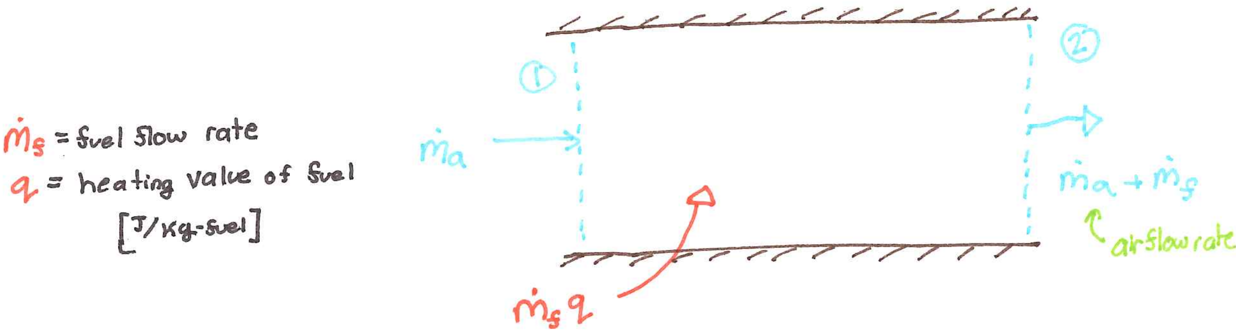


Figure: 1D Rayleigh Flow

1D steady flow equations

Energy Eqn

$$C_p T_1 + \frac{u_1^2}{2} + q = C_p T_2 + \frac{u_2^2}{2} \tag{4.68}$$

$$\dot{m}_a h_{o1} + \dot{m} q = (\dot{m}_a + \dot{m}_f) h_{o2} \tag{4.69}$$

Define f to be the Fuel to Air ratio. Fuel to air ration for a typical gas turbine engine is about 0.02%.

$$f \equiv \frac{\dot{m}_f}{\dot{m}_a} \ll 1 \tag{4.70}$$

Then

$\dot{m}_f q = -C_p T_{o1} \dot{m}_a + (\dot{m}_f + \dot{m}_a) C_p T_{o2}$

$= C_p((\dot{m}_f + \dot{m}_a) T_{o2} - \dot{m}_a (T_{o1}))$

$$C_p T_{o1} + \frac{\dot{m}_f}{\dot{m}_a} q = \left(\frac{\dot{m}_f + \dot{m}_a}{\dot{m}_a}\right) C_p T_{o2} \tag{4.71}$$

$$C_p T_{o1} + f q = (1 + f) C_p T_{o2} \approx C_p T_{o2} \tag{4.72}$$

so if, $C_p =$ constant. Then...

$$T_{o2} = T_{o1} + \frac{f q}{C_p} \tag{4.73}$$

Addition of Heat directly increases total temperature of the working fluid

Property Ratios:

Conservation of Momentum

$$P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2$$

$$\rho u^2 = \rho a^2 M^2 = \rho \frac{\gamma P}{\rho} M^2 = \gamma P M^2 \tag{4.74}$$

Pressure change from state 1 to state 2

$$P_2 - P_1 = \rho_1 u_1^2 - \rho_2 u_2^2 = \gamma P_1 M_1^2 - \gamma P_2 M_2^2 \tag{4.75}$$

Finally, the ratio formed of P2/P1 is

$$\frac{P_2}{P_1} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2} \tag{4.76}$$

The static temperature ratio is

$$\frac{T_2}{T_1} = \frac{P_2}{P_1}\frac{\rho_2}{\rho_1} = \frac{P_2}{P_1}\frac{u_2}{u_1} \tag{4.77}$$

The velocity ratio is

$$\frac{u_2}{u_1} = \frac{M_2 a_2}{M_1 a_1} = \left(\frac{M_2}{M_1}\right)\left(\frac{T_2}{T_1}\right)^2 \tag{4.78}$$

Now take 4.78 and 4.76 into 4.77

$$\frac{T_2}{T_1} = \left(\frac{1 + \gamma M_1^2}{1 + \gamma M_2^2}\right)\left(\frac{M_2}{M_1}\right)^2 \tag{4.79}$$

Finally we have the state density ratio

$$\frac{\rho_2}{\rho_1} = \frac{P_2}{P_1}\frac{T_1}{T_2} \tag{4.80}$$

We must know the mach number in order for these ratios to be found.

Another property we may have interest in is the total pressure ratio

Take 4.76 and 4.14 and we get...

$$\frac{P_{o2}}{P_{o1}} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2}\left(\frac{1 + \frac{\gamma-1}{2} M_2^2}{1 + \frac{\gamma-1}{2} M_1^2}\right)^{\frac{\gamma}{\gamma-1}} \tag{4.81}$$

Finally, 4.77 with 4.13, we get the total temperature ratio

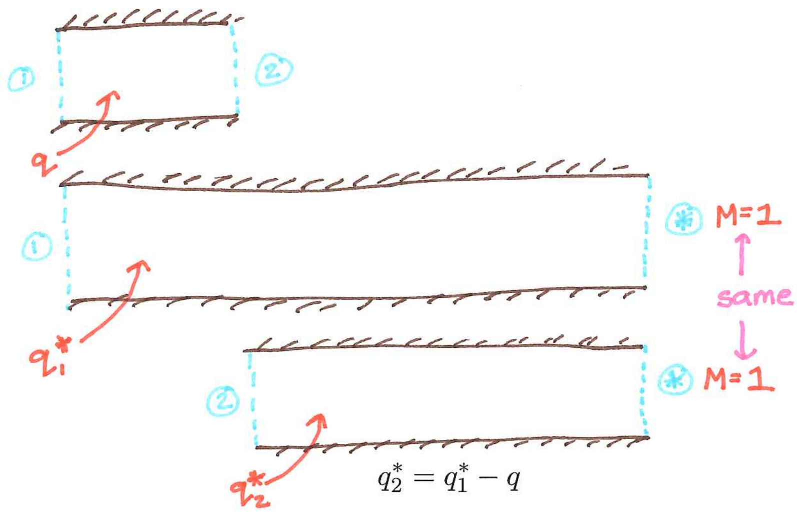
$$\frac{T_{o2}}{T_{o1}} = \left(\frac{1 + \gamma M_1^2}{1 + \gamma M_2^2}\right)^2 \left(\frac{M_2}{M_1}\right)^2 \left(\frac{1 + \frac{\gamma-1}{2} M_2^2}{1 + \frac{\gamma-1}{2} M_1^2}\right) \tag{4.82}$$

Typical Problem: How to get upstream state 2, from downstream state 1?

1. Option 1: Use (4.82). Know M_1, Know T_{o1}, can get T_{o2} from (4.73), solve (4.82) for M_2, then use it in all the other property ratio equations.

2. Option 2: Define a new state, call it the *state (but it's NOT the same as section IV. A. !). Define this new state as the state that results when heat is added or removed to bring the flow to Mach 1.

Graphically:



$$q_2^* = q_1^* - q \tag{4.83}$$

Figure: 1D Rayleigh Flow with *State

If the * condition/state is where $M = 1$, then let $M_1 = 1$ in property ratio equations (4.76), (4.79)(4.81), (4.82) to get:

$$\frac{P}{P^*} = \frac{1+\gamma}{1+\gamma M^2} \tag{4.84}$$

$$\frac{T}{T^*} = M^2 \left(\frac{1+\gamma}{1+\gamma M^2}\right)^2 \tag{4.85}$$

$$\frac{\rho}{\rho^*} = \frac{1}{M^2} \left(\frac{1+\gamma M^2}{1+\gamma}\right) \tag{4.86}$$

$$\frac{P_o}{P_o^*} = \frac{1+\gamma}{1+\gamma M^2} \left(\frac{2+(\gamma-1)M^2}{1+\gamma}\right)^{\frac{\gamma}{\gamma-1}} \tag{4.87}$$

$$\frac{T_o}{T_o^*} = \frac{(\gamma+1)M^2}{(1+\gamma M^2)^2} \left(2+(\gamma-1)M^2\right) \tag{4.88}$$

These are the equations tabulated in Anderson, handout 1d.

Problem Procedure

1. Calculate T_{o2} from (4.73)

2. Calculate property ratios 1 to * via 4.84 → 4.88 w/ M_1

3. Get M_2 from via 4.88 yields M_2

$$\frac{T_{o2}}{T_o^*} = \frac{T_{o2}}{T_{o1}} \frac{T_{o1}}{T_o^*}$$

4. Calculate property ratios 2 to * via 4.84 → 4.88 w/ M_2

5. Solve for properties at state 2,

$$P_2 = \frac{P_2}{P^*} \frac{P^*}{P_1} P_1$$

4.6 Fanno Flow

Fanno flow is 1D flow with friction, now wall shear stress remains in momentum equation.

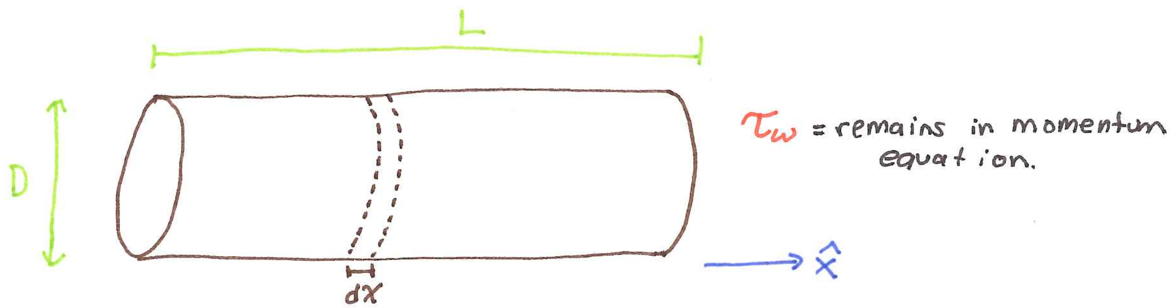


Figure: 1D Fanno Flow

Now shear stress remains in the momentum equation. We are interested in how it behaves in a small segment and then over the entire length.

Momentum equation is then:

$$[(P_2 - P_1) + (\rho_2 u_2^2 - \rho_1 u_1^2)]A = -\int\int_s \tau_w \, \mathrm{d}s = -\int_0^L \pi D \tau_w \mathrm{d}x \tag{4.89}$$

Divide the Area

$$(P_2 - P_1) + (\rho_2 u_2^2 - \rho_1 u_1^2) = -\frac{4}{D}\int_0^L \tau_w \mathrm{d}x \tag{4.90}$$

Now only consider the small element dx, then (4.90) can be written:

$$\mathrm{d}P + \mathrm{d}(\rho u^2) = -\frac{4}{D}\tau_w \, \mathrm{d}x \tag{4.91}$$

Let τ_w be expressed using friction coefficient, **f**, (this is NOT the fuel-air ratio!), as:

$$\tau_w = \frac{1}{2}\rho u^2 f \tag{4.92}$$

Then

$$\mathrm{d}P + \rho u \, \mathrm{d}u + u \, \mathrm{d}(\rho u) = -\frac{1}{2}\rho u^2 \frac{4 f \, \mathrm{d}x}{D} \tag{4.93}$$

Converting (4.93) to Mach# relation →

$$\frac{4\,f\,\mathrm{d}x}{D} = \frac{2}{\gamma\,M^2}\,(1-M^2)(1+\frac{1}{2}(\gamma-1)M^2)^{-1}\,\frac{\mathrm{d}M}{M} \tag{4.94}$$

Integrate between x_1, M_1 and x_2, M_2 to get

$$\int_{x_1}^{x_2}\frac{4\,f\,\mathrm{d}x}{D} = \left[\frac{-1}{\gamma\,M^2} - \frac{\gamma+1}{2\gamma}\ln\left(\frac{M^2}{1+\frac{\gamma-1}{2}\,M^2}\right)\right]_{M_1}^{M_2} \tag{4.95}$$

Flow is adiabatic (but not reversible!), so $T_{o2} = T_{o1}$. Adiabatic but not reversible so pressure will decreases, temperature stays the same because entropy goes up

Then:

Temperature Ratio

$$\frac{T_2}{T_1} = \frac{\frac{T_o}{T_1}}{\frac{T_o}{T_2}} = \frac{2+(\gamma-1)\,M_1^2}{2+(\gamma-1)\,M_2^2} \tag{4.96}$$

Pressure Ratio

$$\frac{P_2}{P_1} = \frac{M_1}{M_2}\sqrt{\frac{T_2}{T_1}} \tag{4.97}$$

Density Ratio

$$\frac{\rho_2}{\rho_1} = \frac{P_2\,T_1}{P_1\,T_2} \tag{4.98}$$

Total Pressure Ratio

$$\frac{P_{o2}}{P_{o1}} = \frac{\frac{P_{o2}}{T_2}\,P_2}{\frac{P_{o1}}{P_1}\,P_1} = \frac{M_1}{M_2}\left[\frac{2+(\gamma-1)\,M_2^2}{2+(\gamma-1)\,M_1^2}\right]^{\frac{\gamma+1}{2(\gamma-1)}} \tag{4.99}$$

Typical Problem: How to get state 2 if you know state 1?

1. Option 1: use (4.95) to solve for M_2, then use property ratios to get other properties, T_2, P_2, etc.

2. Option 2: Define a new state, call it the *state (but it's NOT the same as section IV. A. or section IV. E !). Define this new state as the state that results when friction brings the flow to Mach 1.

Again, let state 1 be this * star state, so the property ratio equations (4.96) to (4.99) become:

$$\frac{T}{T^*} = \frac{\gamma+1}{2+(\gamma-1)\,M^2} \tag{4.100}$$

$$\frac{P}{P^*} = \frac{1}{M}\left[\frac{\gamma+1}{2+(\gamma-1)\,M^2}\right]^{\frac{1}{2}} \tag{4.101}$$

$$\frac{\rho}{\rho^*} = \frac{1}{M}\left[\frac{2+(\gamma-1)\,M^2}{\gamma+1}\right]^{\frac{1}{2}} \tag{4.102}$$

$$\frac{P_o}{P_o^*} = \frac{1}{M}\left[\frac{2+(\gamma-1)\,M^2}{\gamma+1}\right]^{\frac{\gamma+1}{2\,(\gamma-1)}} \tag{4.103}$$

And from (4.95)

$$\frac{4\,\bar{f}\,L^*}{D} = \frac{1-M^2}{\gamma\,M^2} + \frac{\gamma+1}{2\,\gamma}\ln\left[\frac{(\gamma+1)\,M^2}{2+(\gamma-1)\,M^2}\right] \tag{4.104}$$

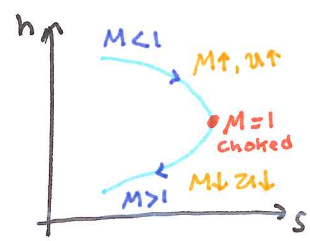
Recognize that $\bar{f}$ is the average friction coefficient over the length L.

These equations (4.100) to (4.104) are tabulated in Anderson tables, handout 1d.

Analysis/solution, similar to Rayleigh flow where now:

$$L_2^* = L_1^* - L$$

4.6.1 Fanna Flow Explained



Continuity

$$\mathrm{d}(\rho u) = 0$$

$$\mathrm{d}(\rho u) = u\mathrm{d}\rho + \rho\,\mathrm{d}u = 0 \quad =$$

Now lets look at enthalpy

$$T\mathrm{d}s = \mathrm{d}h - \frac{\mathrm{d}p}{\rho}$$
$$h_o = h + \frac{u^2}{2}$$

$$\mathrm{d}h_o = \mathrm{d}h + u\mathrm{d}u = 0 \qquad \text{Adiabatic} \tag{b}$$

$$T\mathrm{d}s = -u\,\mathrm{d}u - \frac{\mathrm{d}\rho}{\rho}$$

$$dp = -\rho\, u\, du - \rho\, T\, ds \tag{c}$$

$$a^2 = \left.\frac{dp}{d\rho}\right)_s$$

(b)(c) and (4.6) into (a)

$$d(\rho\, u) = \left[\frac{1}{a^2}\, dp + \frac{d\rho}{ds}\Big|_p\, ds\right] u + \rho du = 0$$

$$= \left(1 - \frac{u^2}{a^2}\right) \rho du - \left[\frac{\rho\, T}{a^2} - \left(\frac{\partial \rho}{\partial s}\right)_p\right] u\, ds = 0$$

underscore P means constant pressure

Move it to the other side so that

$$\left(1 - \frac{u^2}{a^2}\right) \rho\, \frac{du}{dx} = \left[\frac{\rho\, T}{a^2} - \left(\frac{\partial \rho}{\partial s}\right)_p\right] u\, \frac{ds}{dx} \tag{d}$$

We are interested in how u, the velocity of the fluid moving through the channel, changes along the duct with respect to the position.

This relation ship is showing us how entropy change over the length of the duct which related to the velocity with respect the length.

If entropy goes up, density goes down. see eqn 2.26

$$\frac{\rho T}{a^2} > 0 \qquad -\left(\frac{\partial \rho}{\partial s}\right)_P > 0$$

Since the RHS is positive, LHS must be positive.

If $u < a$ (M<1) then

The velocity of the fluid must accelerate

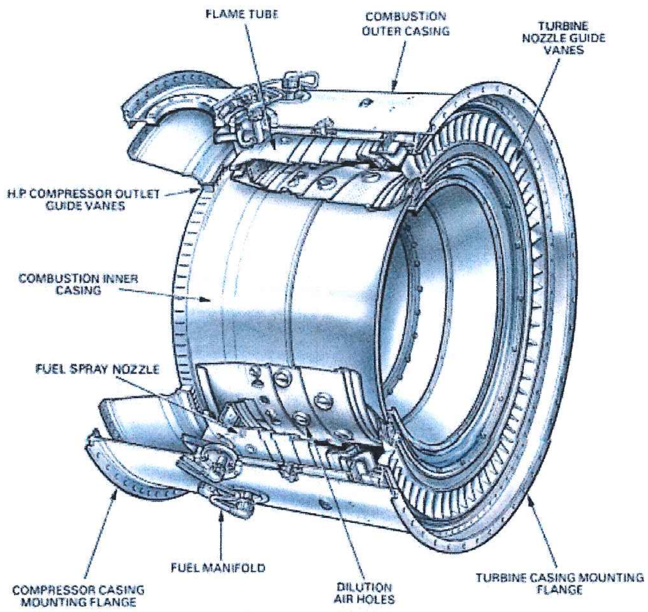
$$\frac{\partial u}{\partial x} > 0$$

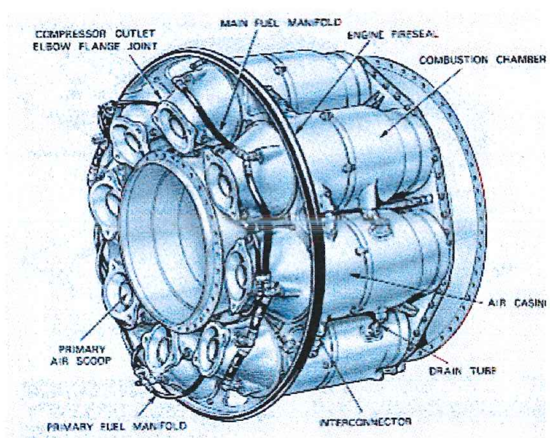
If $u > a$ (M>1) then

$$\frac{\partial u}{\partial x} < 0$$

4.7 Sample Burner Can Calculation

Fluid coming in from the left. coming out on the outer ring. This burner can is annulus.





Burner can may not be a circle (can) cross section. Take hydraulic diameter for analysis:

$$D = \frac{4A}{P} \tag{4.105}$$

Where A = Cross Sectional Area $[m^2]$ and P = wetted perimeter [m]

Aircraft at 39,000 ft, M = 0.85, compressor ratio of 10:1 Inlet state to the burner can is state 2, estimate the properties of the flow coming out of the burner can, call it state 4.

Assume we know:

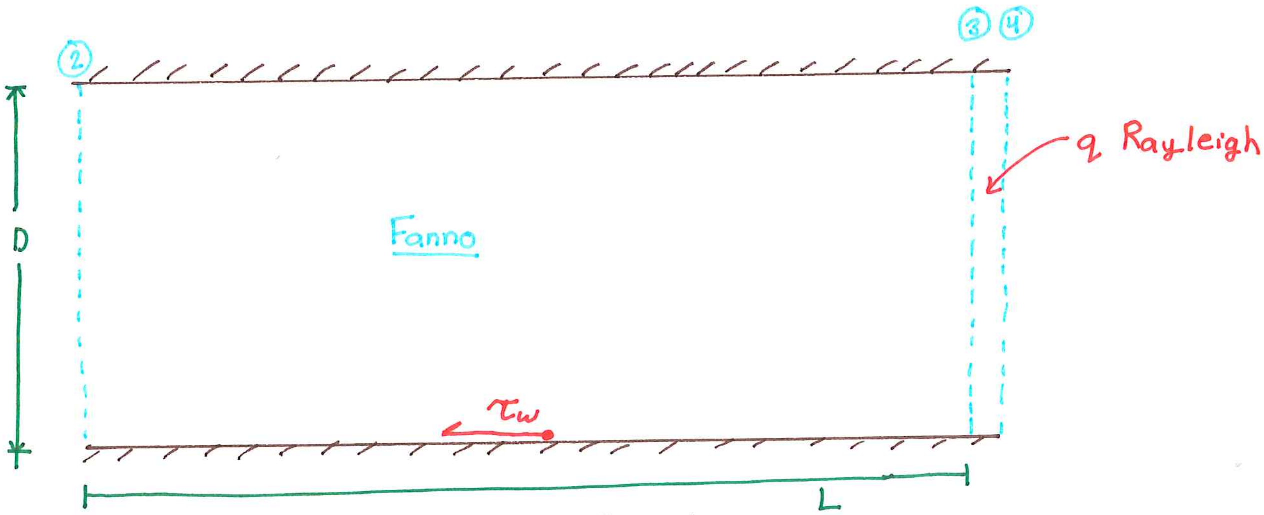
P2 = 5 atm	M2 =0.16	Gamma = 1.4
D = 0.36m	Burner efficiency = 98%	T2 = 483 K
L =0.60 m	q =43,000 kJ/kg-fuel	f (fuel-air) =0.025
f (friction) = 0.60	Cp3 = 1050 J/kg-K	Cp4 = 1150 J/kg

We assume CP changes a bit due to heat addition. Notice friction and heat addition being done at the same time as the fluid flows through. Its an integrated coupled interaction that we doesn't take both.

Heat addition and friction are both occurring at the same time and same locations within the combustor/burner can.

An integrated solution would require numerical techniques. Here, we can estimate the effects of friction and heat addition by breaking the processes up.

State 2 to 3, Friction only State 3 to 4 Heat addition only



Decoupled Burner Can Diagram

We are assuming we can break up these effects (really we cant, but lets see). Relatively simple and analytical model for how properties change.

Fanno flow (state 2 to 3)

Calculate total condition at $M_2 = 0.16$

$$P_{o2} = P_2\,1.013 = 5.07\,\text{atm}$$
$$T_{o2} = T_2\,1.005 = 485.4\,\text{K}$$

Fanno Flow Tables (Anderson A.4) @ $M_2 = 0.16$

$$\frac{4fL^*}{D} = 24.20 \qquad \frac{P_2}{P^*} = 6.829 \qquad \frac{P_{o2}}{P_o^*} = 3.673 \qquad \frac{T_2}{T^*} = 1.194 \qquad L = 0.6[m]$$

$$L_3^* = L_2^* - L$$

$$\frac{4fL_3^*}{D} = \frac{4fL_2^*}{D} - \frac{4fL}{D} = 24.20 - \frac{4 \cdot 0.6 \cdot 0.6}{0.36} = 20.20$$

for $\frac{4fL_3^*}{D} = 20.20$ therefore $M_3 = 0.174$

$$\frac{P_3}{P^*} = 6.29 \qquad \frac{T_3}{T^*} = 1.193$$
$$P_3 = 4.605 \qquad T_3 = 482.6$$
$$P_{o3} = 4.685 \qquad T_{o3} = T_{o2} = 485.4\,[K]$$

Rayleigh flow (state 3 to 4)

Anderson A.3 when $M_3 = 0.174$

$$\frac{P_3}{P^*} = 2.302 \qquad \frac{P_{o3}}{P_o^*} = 1.243 \qquad \frac{T_3}{T^*} = 0.161 \qquad \frac{T_{o3}}{T_o^*} = 0.135$$

$$T_{o4} = \frac{C_{p3}\,T_{o3} - f\,\eta_B\,q}{C_{p4}\,(1+f)} = 1335\,K$$

$$\frac{T_{o4}}{T_o^*} = \frac{T_{o4}}{T_{o3}}\frac{T_{o3}}{T_o^*} = 0.371 \rightarrow M_4 = 0.313$$

$$\frac{P_4}{P^*} = 2.110 \qquad \frac{P_{o4}}{P_o^*} = 1.193 \qquad \frac{T_4}{T^*} = 0.436$$

$$P_4 = 4.221\,[\text{atm}]$$
$$P_{o4} = 4.497\,[\text{atm}]$$

Total Pressure Ratio

$$\pi_B = \frac{P_{o4}}{P_{o2}} = \frac{P_{o3}}{P_{o2}}\frac{P_{o4}}{P_{o3}} = 0.887$$

We want this to be low because there is more entropy change. We want this to be as close to one as possible. We see that the total pressure drop is by around 12% for our model. Performance is a good parameter to keep track of

Not all of the fuel combust because of various inefficiency.

Total pressure ratio = the measure of burner efficiency want $\pi_B = 1$ cause ΔP_o related to Δs. Δs increases and ΔP_o decreases.

4.8 Variable Area (Nozzle) Flow

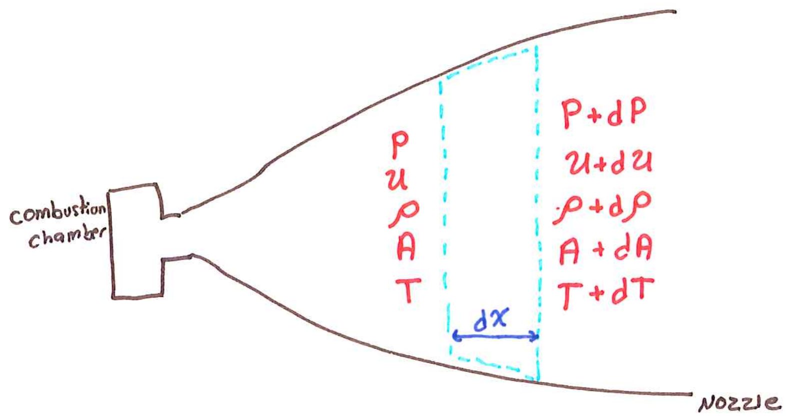


Figure: Nozzle Flow Diagram

How does varying area affect the flow and the state variables?

Continuity

$$\dot{m} = \rho\, u\, A = const. \rightarrow \mathrm{d}(\rho u A) = const.$$

$$\frac{\mathrm{d}\rho}{\rho} + \frac{\mathrm{d}u}{u} + \frac{\mathrm{d}A}{A} = 0 \tag{4.106}$$

Then

$$\frac{\mathrm{d}A}{A} = (M^2 - 1)\frac{\mathrm{d}u}{u} \tag{4.107}$$

If $M < 1$

- Decreasing Area = Increasing Velocity
- Increasing Area = Decreasing Velocity.

For $M >$

- Decreasing Area = Decreasing Velocity
- Increasing Area = Increasing Velocity

So to go from $M < 1$ to $M > 1$ need converging-diverging shape, i.e., a nozzle.

Figure: Converging Diverging Nozzle

Consider arbitrary point (2) in downstream diverging section, what are M_2, P_2, T_2 if know A_2 and star* conditions?

Continuity:

$$\rho^* \, u^* \, A^* = \rho_2 \, u_2 \, A_2 \tag{4.108}$$

Assume isentropic nozzle flow:

$$\frac{A}{A^*} = \frac{\rho^*}{\rho} \frac{a^*}{u} = \frac{\rho^*}{\rho_o} \frac{\rho_o}{\rho} \frac{a^*}{u} \tag{4.109}$$

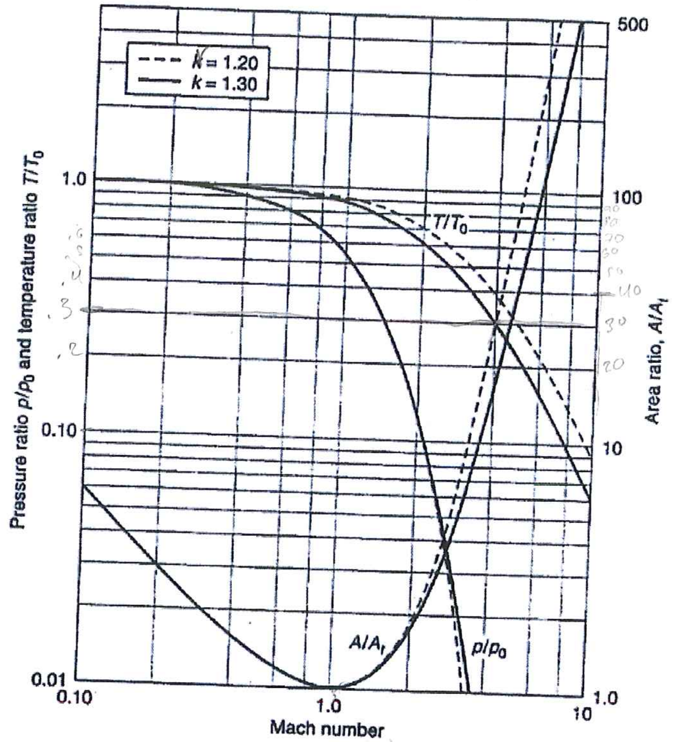
Plug in (4.15) and (4.24) and we get

$$\frac{A}{A^*} = \frac{1}{M^2} \left[\frac{2}{\gamma + 1} \left(1 + \frac{\gamma - 1}{2} M^2 \right) \right]^{\frac{\gamma+1}{2(\gamma-1)}} \tag{4.110}$$

Mach # at any location in the nozzle is a function of A/A*, the area ratio.

k in this figure is gamma (ratio of specific heats), k is the old rocket nomenclature? Notice in this figure k is not 1.4, because rockets rarely, never, have $\gamma = 1.4$. Usually less.

3.2. SUMMARY OF THERMODYNAMIC RELATIONS



Other properties related to known throat * condition.

For Pressure

$$P_2 = \frac{P_2}{P_o}\frac{P_o}{P^*}P^* \tag{4.111}$$

For Temperature

$$T_2 = \frac{T_2}{T_o}\frac{T_o}{T^*}T^* \tag{4.112}$$

How does this varying area affect the total pressure and total temperature.

Total pressure and temperature do not change because this is an isentropic flow which also means it is adiabatic. There is a maximum amount of mass (really maximum amount of entropy) that you can force through a given area size.

Choked (Max) Mass Flow Rate

$$\dot{m} = \frac{P_o A^*}{\sqrt{T_o}}\sqrt{\frac{\gamma}{R}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}} \tag{4.113}$$

$\frac{\dot{m}\sqrt{T_o R}}{P_o A^*} = 0.6847$ for $\gamma = 1.4$

Section 1: Fundamentals

AE435

Spring 2018

5 Combustion

Here we will develop models for predicting the **equilibrium composition of a gas** at high temperature. And we will develop a model for predicting the temperature of that gas, assuming an adiabatic process, a.k.a., the **adiabatic flame temperature**.

Contents

5.1 Introduction Concepts

Figure:

Combustion is important because its how heat addition is introduced into the flow. Energy goes into these systems via the chemical bond energy in the propellant. That energy gets released via combustion that can then be converted into high enthalpy gas that gets accelerated at the end of the turbine.

5.1.1 General Chemical Reaction

$$\nu_A\, A + \nu_B\, B + ... \rightarrow \nu_P\, P + \nu_Q\, Q + ... \tag{5.1}$$

Where

$$\nu_i = \text{Molar Stoichiometric Coeffieicnt}$$

Examples:

$$\begin{aligned} H_2 + O & \qquad 2\,H_2 + O_2 \rightarrow 2\,H_2O \\ \text{Methane} + O_2 & \qquad CH_4 + 2\,O_2 \rightarrow CO_2 + 2\,H_2O \\ \text{Octane} + O_2 & \qquad C_8H_{18} + \frac{25}{2}\,O_2 \rightarrow 8\,CO_2 + 9\,H_2O \end{aligned} \tag{5.2}$$

5.1.2 Mixtures

Air, Dry Air

$$N_2 : n_{N_2} = 0.781$$

$$O_2 : n_{O_2} = 0.209$$

$$Ar : n_{Ar} = 0.010$$

Where

$$n_i = \text{mole fraction}$$

so now...

Effective Molecular Weight

$$\text{Effective MW} = \Sigma\, n_i\, MW_i \tag{5.3}$$

We use a mole fraction when we are trying to calculate a property thats on a per-mole bases

N2 O2 Ar

$$\Sigma\, n_i MW_i = (0.781) * 28 + 0.209 * 32 + 0.01 * 40 = 28.96 \frac{kg}{kmol}$$

a mass fraction is referenced by μ_i. If you are calculating a property that is per mass basis, you will use mass fraction. IE.

$$N_2 : \mu_{N_2} = 0.755$$

$$O_2 : \mu_{O_2} = 0.231$$

$$Ar : \mu_{Ar} = 0.014$$

If you wanted to calculate C_P then

For Air:

$$C_{Pmix} = \Sigma_i\, \mu\, C_{Pi} \qquad R_{mix} = \Sigma_i\, \mu_i\, R_i$$

$$C_{vmix} = \Sigma_i\, \mu\, C_{vi} \qquad R_{mix} = \frac{R}{MW_{mix}} = \frac{8314}{28.96} = 287 \frac{J}{kg - K} \tag{5.4}$$

Since Argon is inert and in small quantities, it's often neglected, then:

$$\left.\begin{matrix} 21\%\, O_2 \\ 79\%\, N_2 \end{matrix}\right\} \quad \text{new definition for air with } MW = 28.01 \frac{kg}{kmol}$$

For each mole of O_2, we have:

OLD Definition

$$\frac{\nu_{N_2}}{\nu_{O_2}} = \frac{0.781}{0.209} = 3.74\,\text{moles of } N_2$$

NEW Definition

$$\frac{\nu_{N_2}}{\nu_{O_2}} = \frac{0.79}{0.21} = 3.76\,\text{moles of } N_2$$

Now hydrogen combustion with air:

$$2\,H_2 + O_2 + 2.76\,N_2 \rightarrow 2H_2O + 3.76\,N_2$$

Octane combustion with air:

$$C_8H_18 + \frac{25}{2}(O_2 + 3.76\,N_2) \rightarrow 8\,CO_2 + 9H_2O + \frac{25}{2}(3.76\,N_2)$$

NOTICE there is no oxygen of hydrogen the products side. All has combusted and none is left over.

5.1.3 Stoichiometric

These have all so far been stoichiometric reactions.

Stoichiometric Reaction - a reaction wherein no fuel or oxidizer are left over as products (perfectly matched, all fuel and oxidizer consumed in the reaction).

This is rarely the case in rockets and gas turbine engines. Stoichiometric temperatures are just to darn high. We would melt the jet engines.

$$\begin{aligned} &\text{Not Stoichiometric:} \\ &\qquad 2\,H_2 + 2\,O_2 \rightarrow 2\,H_2O + O_2 \\ &\text{Stoichiometric:} \\ &\qquad 2\,H_2 + O_2 + O_2 \rightarrow 2\,H_2O + O_2 \end{aligned} \tag{5.5}$$

We define how far from stoichiometric an actual reaction is with the equivalence ratio.

$$\phi = \frac{\left(\frac{n_{fuel}}{n_{oxidizer}}\right)_{actual}}{\left(\frac{n_{fuel}}{n_{oxidizer}}\right)_{stoich}} \tag{5.6}$$

Where n = the number of moles

We see that if

$\phi = 1$	Stoichiometric
$\phi > 1$	Fuel Rich, excess fuel
$\phi < 1$	Fuel Lean, excess oxidizer

5.2 Elementary Chemical Reaction Kinetics

So far we have been doing global chemical reactions when in reality, there are many steps that happen that lead to the final state of the reactions. IT is important to know what these are because they lead to the final reaction of the mixture.

Imagine a volume of air. We add some H_2 fuel, and the mixture combusts with global chemical reaction:

$$2\,H_2 + O_2 + 3.76\,N_2 \rightarrow 2\,H_2O + 3.76\,N_2 \tag{5.7}$$

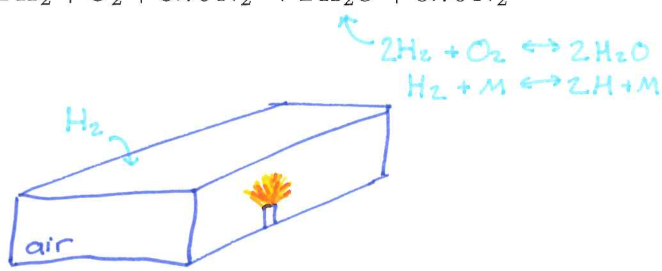


Figure: Chemical Reaction Apparatus

But?.. Actually there are many elementary or "building block" reactions occurring within the combustion volume. Hydrogen H_2 just doesn't collide with O_2 and poof out pops water. No, there are many subreactions that take place to get to water on the other side?

For example:

Dissociation Reactions

$$H_2 + h\,\nu \rightarrow 2\,H$$

$$O_2 + h\nu \rightarrow O_2 \tag{5.8}$$

$$N_2 + h\nu \rightarrow 2N$$

$$2H + O \rightarrow H + OH$$

$$H + O_2 \rightarrow O + OH$$

3-Body Recombination

$$H + H + M \rightarrow H_2 + M$$

$$O + O + M \rightarrow O_2 + M$$

$$N + N + M \rightarrow N_2 + M \tag{5.9}$$

$$N + O + M \rightarrow NO + M$$

$$H + OH + M \rightarrow H_2O + M$$

How hot is the product gas going to be? If we want to predict the equilibrium composition and the temperature we must understand the final reactions.

In general, one can write an elementary reaction as:

$$A + B \rightarrow C + D \tag{5.10}$$

Classical model for the collision that leads to the chemical reaction is:

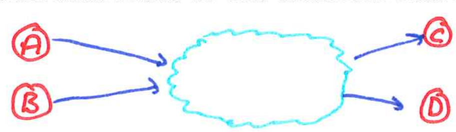


Figure: Billiard Ball Model

However, before this reaction occurs, certain things must happen:

1. The reactants must have a **COLLISION**.

Figure: Collision

2. The internal energy (translation, vib, rot, elec) of the reactants must be above a certain **Threshold/Activiation Energy**.
3. The reacting particles must have the right **ORIENTATION** (i.e., Steric factor).

Figure: Orientation

5.3 Law of Mass Action

The law of mass action allows us to develop rate equations for the elementary reactions in a combustion reaction.

The rate of an elementary reaction is proportional to the product of the concentrations of participating species.

Consider an elementary combustion reaction:

$$H_2 + M \xrightarrow{k_f} 2H + M \tag{5.11}$$

If we put H_2 and M into a volume at t = 0 , after a time τ there is now H_2, M, and H in the box. Some of the H_2 has dissociated.

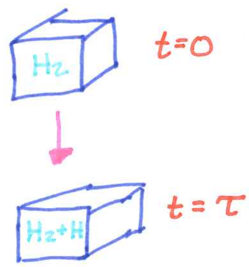


Figure: Dissasociated Control Volume

Actually, the reverse reaction occurs as well.

$$2H + M \xrightarrow{k_b} H_2 + M \tag{5.12}$$

We write (5.11) and (5.12) together as:

$$H_2 + M \underset{k_b}{\overset{k_f}{\longleftrightarrow}} 2\,H + M \tag{5.13}$$

Where k_f and k_b are the forward and backward reaction rate coefficients.

For a general elementary reaction:

$$A + B \underset{k_b}{\overset{k_f}{\longleftrightarrow}} C + D \tag{5.14}$$

the law of mass action allows us to write:

$$\frac{\mathrm{d}(A)}{\mathrm{d}t} = -k_f\,(A)\,(B) + k_b\,(C)\,(D)$$

(A) is the molar concentration of species A. $\frac{\#\text{ of moles}}{\text{volume}}$

We have $-k_f$ the forward reactions, which destroys A and reduces the concentration of A where as the backwards is creating species A and is increasing the concentration with time.

Also,

$$\frac{\mathrm{d}(B)}{\mathrm{d}t} = -k_f\,(A)\,(B) + k_b\,(C)\,(D)$$
$$\frac{\mathrm{d}(C)}{\mathrm{d}t} = k_f\,(A)\,(B) - k_b\,(C)\,(D) \tag{5.15}$$
$$\frac{\mathrm{d}(D)}{\mathrm{d}t} = k_f\,(A)\,(B) - k_b\,(C)\,(D)$$

k_f and k_b have a mathematical form (called Arrhenius form, 1889), based on a billiard ball model

[Rate of Reaction] = [Rate of Collisions] x [Fraction with Enough Energy] x [Fraction with Right Orientation]

$$= CT^{\frac{1}{2}} \exp\left(-\frac{E}{RT}\right) \times \text{Steric Factor}$$

$$k \propto ZT^n \exp\left(-\frac{E}{RT}\right)$$

Where

E = Activation Energy

n ≈ 0.5.

5.4 Chemical Equilibrium Composition

We are usually interested in the composition of the combustion reaction when it reaches equilibrium, that is, when it is not changing anymore with time.

Consider H_2 dissociation:

$$M + H_2 \leftrightarrow 2H + M \tag{5.16}$$

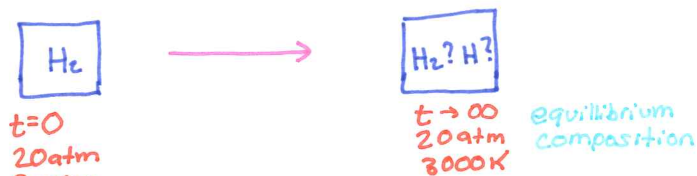


Figure: Equilibrium Composition

What happens is some H_2 disassociates, some does not.

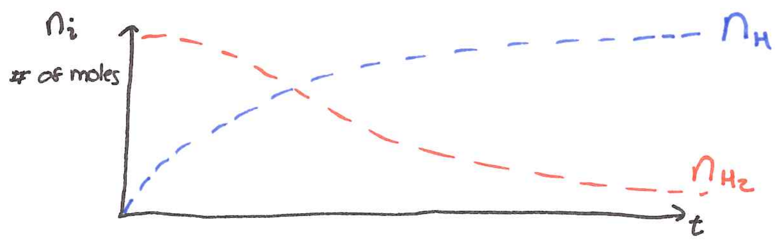


Figure: Temporal Relation of disassociation

For:

$$A + B \overset{k_f}{\underset{k_b}{\longleftrightarrow}} C + D$$

At equilibrium, the rate of change of the species is zero with time.

This is:

$$\frac{\mathrm{d}(A)}{\mathrm{d}t} = -k_f\,(A)\,(B) + k_b\,(C)\,(D) \tag{5.17}$$

Equilibrium Constant with respect to Molar Concentrations

$$\frac{k_f}{k_b} = \frac{(C)(D)}{(A)(B)} = k_c \tag{5.18}$$

For H_2 Dissociation:

At $t \rightarrow \infty$

$$\frac{\mathrm{d}H_2}{\mathrm{d}t} = -k_f(H_2)(M) + k_b(H)(H)(M) = 0 \tag{5.19}$$

$$\frac{k_f}{k_b} = k_c = \frac{(H)^2}{(H_2)} \tag{5.20}$$

Notice that

$$\frac{\mathrm{d}(H)}{\mathrm{d}T} = 2\left[k_f(H_2)(M) - k_b(H)(H)(M)\right] = 0 \tag{5.21}$$

$$\frac{k_f}{k_b} = k_c = \frac{(H)^2}{(H_2)} \tag{5.22}$$

In general:

$$\nu_A A + \nu_B B + ... \rightarrow \nu_C C + \nu_D D + ... \tag{5.23}$$

We have written our equilibrium constant, k_c, with respect to concentration. It is more convenient to define the equilibrium rate constant k_p in terms of partial pressure, that is:

$$k_p = \frac{P_C^{\nu_C} P_D^{\nu_D} \cdots}{P_A^{\nu_A} P_B^{\nu_B} \cdots} \tag{5.24}$$

Where $p_i = n_i p$ where p is total pressure in the vessel, pi is the partial pressure of species (i) and n_i is the mole fraction of (i).

Then $k_p =$

$$k_p = \frac{(\nu_C P)^{\nu_C} (\nu_D P)^{\nu_D} ...}{(\nu_A P)^{\nu_A} (\nu_B P)^{\nu_B} ...} = \left[\frac{n_C^{\nu_C} n_D^{\nu_D} \cdots}{n_A^{\nu_A} n_B^{\nu_B} \cdots}\right] P^{(\nu_C + \nu_D + ...) - (\nu_A + \nu_B + ...)} \tag{5.25}$$

k_p made dimensionless by dividing by a reference pressure, defined to be $1atm = p_{ref}$, then:

General Equation for Equilibrium Constant

$$K = \left[\frac{n_C^{\nu_C} n_D^{\nu_D} \cdots}{n_A^{\nu_A} n_B^{\nu_B} \cdots}\right] \left(\frac{P}{P_{\mathrm{ref}}}\right)^{(\nu_C + \nu_D + ...) - (\nu_A + \nu_B + ...)} \tag{5.26}$$

These K values given in the handout! Handout 1g, given on COMPASS and below.

Example $H_2 + M \leftrightarrow 2H + M$

We are using 5.26, the general equation for equilibrium constant and we are applying it to this specific hydrogen dissociation.

$$K = \frac{(n_H)^2}{(n_{H2})}\left(\frac{P}{P_{ref}}\right)^{2-1} = \frac{n_H^2}{n_{H2}}\frac{20}{1} \rightarrow K = \frac{n_H^2}{n_{H2}}20$$

We are at a pressure of 20atm wrt the reference pressure which is 1atm. Now use the table to look up K at 3000K

We go to the table and find that it is −3.685 recognize that that is LN of K therefore $K = \exp(-3.685) = 0.025$.

$$\therefore \quad 0.025 = \frac{n_H^2}{n_{H2}} \cdot 20 \longrightarrow \frac{n_H^2}{n_{H2}} = 1.25E{-3}$$

So now you can write....1.25×10^{-3}. We have two unknowns, nH and nH_2. Another equation we have is that the sum of the fractions must equal 1. So that fives us another equation that we can work with. two equation, two unknowns. We end up with a quadratic equation that we can solve.

$$n_H + n_{H2} = 1 \qquad \frac{n_H^2}{1 - n_H} = 1.25E{-3}$$

We find that $nH = 0.0346$ and $nH_2 = 0.9654$. So at that pressure and temperature, we are 3.5% dissociated.

$$n_H = 0.0346$$
$$n_{H2} = 0.9654$$

NOTE: Simply use linear approximation if our K is between the values.

If temperature increased, we should expect to see more dissociation. If pressure were to decrease (say 20 atm - 5 atm) now nH = 6.8%. Pressure goes up, dissociation goes down.

Table A.12
Logarithms to the Base e of the Equilibrium Constant K

These are ln(K) values

For the reaction $\nu_A A + \nu_B B \rightleftharpoons \nu_C C + \nu_D D$ the equilibrium constant K is defined as

$$K = \frac{a_C^{\nu_C} a_D^{\nu_D}}{a_A^{\nu_A} a_B^{\nu_B}}$$

Base on thermodynamic data given in the JANAF Thermochemical Tables, Thermal Research Laboratory, The Dow Chemical Company, Midland, Michigan.

Temp. K	$H_2 \rightleftharpoons 2H$	$O_2 \rightleftharpoons 2O$	$N_2 \rightleftharpoons 2N$	$H_2O \rightleftharpoons H_2 + \frac{1}{2}O_2$	$H_2O \rightleftharpoons \frac{1}{2}H_2 + OH$	$CO_2 \rightleftharpoons CO + \frac{1}{2}O_2$	$\frac{1}{2}N_2 + \frac{1}{2}O_2 \rightleftharpoons NO$
298	−164.005	−186.975	−367.480	−92.208	−106.208	−103.762	−35.052
500	−92.827	−105.630	−213.372	−52.691	−60.281	−57.616	−20.295
1000	−39.803	−45.150	−99.127	−23.163	−26.034	−23.529	−9.388
1200	−30.874	−35.005	−80.011	−18.182	−20.283	−17.871	−7.569
1400	−24.463	−27.742	−66.329	−14.609	−16.099	−13.842	−6.270
1600	−19.637	−22.285	−56.055	−11.921	−13.066	−10.830	−5.294
1800	−15.866	−18.030	−48.051	−9.826	−10.657	−8.497	−4.536
2000	−12.840	−14.622	−41.645	−8.145	−8.728	−6.635	−3.931
2200	−10.353	−11.827	−36.391	−6.768	−7.148	−5.120	−3.433
2400	−8.276	−9.497	−32.011	−5.619	−5.832	−3.860	−3.019
2600	−6.517	−7.521	−28.304	−4.648	−4.719	−2.801	−2.671
2800	−5.002	−5.826	−25.117	−3.812	−3.763	−1.894	−2.372
3000	−3.685	−4.357	−22.359	−3.086	−2.937	−1.111	−2.114
3200	−2.534	−3.072	−19.937	−2.451	−2.212	−0.429	−1.888
3400	−1.516	−1.935	−17.800	−1.891	−1.576	0.169	−1.690
3600	−0.609	−0.926	−15.898	−1.392	−1.088	0.701	−1.513
3800	0.202	−0.019	−14.199	−0.945	−0.501	1.176	−1.356
4000	0.934	0.796	−12.660	−0.542	−0.044	1.599	−1.216
4500	2.486	2.513	−9.414	0.312	0.920	2.490	−0.921
5000	3.725	3.895	−6.807	0.996	1.689	3.197	−0.686
5500	4.743	5.023	−4.666	1.560	2.318	3.771	−0.497
6000	5.590	5.963	−2.865	2.032	2.843	4.245	−0.341

What we did is fine for simple reactions. What about more complex reactions?

For instance,

$$aH_2 + bO_2 \rightarrow \mathscr{N}_{H_2O}\, H_2O + \mathscr{N}_{O_2}\, O_2 + \mathscr{N}_{H_2}\, H_2 + \mathscr{N}_O\, O + \mathscr{N}_H\, H + \mathscr{N}_{OH}\, OH$$

Now within those products you have multiple reactions occurring.

1. $H_2 \longleftrightarrow 2H$
2. $O_2 \longleftrightarrow 2O$
3. $H_2O \longleftrightarrow H_2 + \frac{1}{2}O_2$
4. $H_2O \longleftrightarrow \frac{1}{2}H + OH$

How do we figure this out?

1. First Balance it Stoichiometrically. Conservation of Atoms.

$$H: \qquad 2a = 2\mathscr{N}_{H_2O} + 2\mathscr{N}_{H_2} + \mathscr{N}_H + \mathscr{N}_{OH}$$

$$O: \qquad 2b = \mathscr{N}_{H_2O} + 2\mathscr{N}_{O_2} + \mathscr{N}_O + \mathscr{N}_{OH}$$

2. Find the Equilibrium constant for each of these reactions

$$K = \frac{\left(\frac{\mathscr{N}_H}{\mathscr{N}}\right)^2}{\frac{\mathscr{N}_{H_2}}{\mathscr{N}}} \left(\frac{P}{P_{\text{ref}}}\right)^{2-1} \qquad K = \frac{\left(\frac{\mathscr{N}_O}{\mathscr{N}}\right)^2}{\frac{\mathscr{N}_{O_2}}{\mathscr{N}}} \left(\frac{P}{P_{\text{ref}}}\right)^{2-1}$$

$$K = \frac{\left(\frac{\mathscr{N}_{H_2}}{\mathscr{N}}\right)\left(\frac{\mathscr{N}_{O_2}}{\mathscr{N}}\right)^{\frac{1}{2}}}{\frac{\mathscr{N}_{H_2O}}{\mathscr{N}}} \left(\frac{P}{P_{\text{ref}}}\right)^{\frac{3}{2}-1} \qquad K = \frac{\left(\frac{\mathscr{N}_{H_2}}{\mathscr{N}}\right)^{\frac{1}{2}}\left(\frac{\mathscr{N}_{OH}}{\mathscr{N}}\right)}{\frac{\mathscr{N}_{H_2O}}{\mathscr{N}}} \left(\frac{P}{P_{\text{ref}}}\right)^{\frac{3}{2}-1}$$

$$N = \mathscr{N}_{H_2O} + \mathscr{N}_{O_2} + \mathscr{N}_{H_2} + \mathscr{N}_O + \mathscr{N}_H + \mathscr{N}_{OH}$$

3. For known Temperature and Pressure, we have 6 equations and 6 unknowns. Find the total moles

For many models, for air combustion include over 20 different elementary reactions.

5.5 Energy Balance in Chemical Transformations

5.5.1 Enthalpy of Formation (Heat of Formation)

The heat of formation of a substance is the heat interaction that occurs when one mole of substance is formed from its elements as they occur in nature.

This is the enthalpy (or heat or energy) required to form the substance. I think of this as energy stored in the chemical bonds of the substance (not sensible energy). Mathematically we write the enthalpy of formation as:

$$\Delta \bar{h}_f^\circ$$

Mathematically this comes from

Enthalpy of Formation

$$\Delta \bar{h}_f^\circ = \bar{h}_f^\circ - \Sigma_i \nu_i (\Delta \bar{h}_f^\circ)_i \tag{5.27}$$

For example:

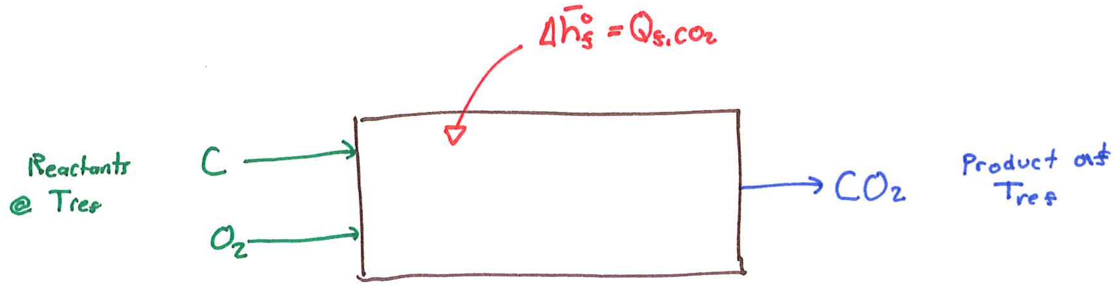


Figure: Enthalpy Reaction Diagram

1. Elements in standard state (298K 1atm) have $\Delta \bar{h}_f^\circ = 0$

$$O_2 \qquad N_2 \qquad H_2 \qquad \rightarrow \Delta \bar{h}_f^\circ = 0$$

2. For compounds, $\Delta \bar{h}_f^\circ$ is tabulated from spectroscopic data (see tables in handout 1h):

$$CO_2 \qquad \Delta \bar{h}_{f,H_2O(g)}^\circ = -94051[cal/mol]$$

$$CH_4 \qquad \Delta \bar{h}_{f,H_2O(g)}^\circ = -17889[cal/mol]$$

Nitrogen, Diatomic (N_2) (9/30/65)	Nitrogen, Monatomic (N) (3/31/61)
$(\bar{h}_f^\circ)_{298} = 0$ kJ/kmol $M = 28.013$	$(\bar{h}_f^\circ)_{298} = 472\ 646$ kJ/kmol $M = 14.007$

Oxygen, Diatomic (O_2) (9/30/65)	Oxygen, Monatomic (O) (6/30/62)
$(\bar{h}_f^\circ)_{298} = 0$ kJ/kmol $M = 31.999$	$(\bar{h}_f^\circ)_{298} = 249\ 195$ kJ/kmol $M = 16.00$

3. H_2O can be either liquid or gas at 298K 1atm, so:

$$\Delta\bar{h}^\circ_{f,H_2O(g)} = -57795 \qquad [cal/mol]$$

$$\Delta\bar{h}^\circ_{f,H_2O(l)} = -68315 \qquad [cal/mol]$$

5.5.2 Enthalpy of Reaction

Keep track of enthalpy (heat/energy) lost during reaction. Energy stored in chemical bonds of reactants get released during the reaction, some of this energy will be used to form the bonds of the product species. Whatever is left over will be available to increase the temperature of the reaction

Enthalpy of Reaction

$$\Delta\bar{h}^\circ_r = \left[\Sigma_i N_i(\Delta\bar{h}^\circ_f)_i\right]_{\text{products}} - \left[\Sigma_i N_i(\Delta\bar{h}^\circ_f)_i\right]_{\text{reactants}} \tag{5.28}$$

EXAMPLE:

$$CH_4 + 2O_2 \longrightarrow CO_2 + 2H_2O(g)$$

Species	CH_4	O_2	CO_2	H_2O
$\Delta\bar{h}^\circ_r$	-17889	0	-94051	-57795

$$\Delta\bar{h}^\circ_r = \left[-94051 + 2(-57795)\right]_{\text{products}} - \left[-17889 + 2(0)\right]_{\text{reactants}}$$

$$= -191,752 \qquad \left[\frac{cal}{mol\,CH_4}\right]$$

Negative therefore this is an exothermic reaction. There is still heat/energy available. Not all of the reactants energy is used to create the products. This extra energy is used to compute the final temperature that the products are in. With this extra energy, how hot are the products going to get?

5.5.3 Energy Equation for Chemical Reactions

Schematically:

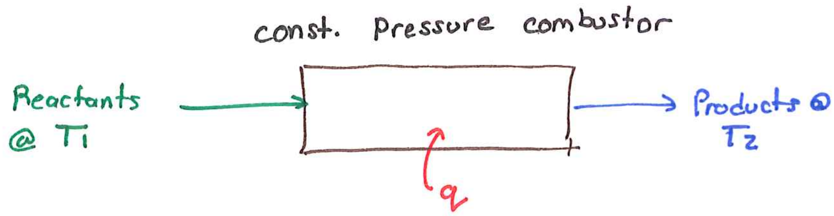


Figure: Schematic Diagram of Chemical Reaction

Energy in = Energy out

We neglect the KE and PE of the species.

What energy goes in: Heat (q), energy stored in bonds via enthalpy of formation. Finally the internall energy which is given by enthalpy at the initial temperature.

$$
{}_1q_2 + \left[\Sigma_i N_i (\Delta \bar{h}_f^\circ)_i\right]_{\text{reactants}} + \left[\Sigma_i n_i (\bar{h}_1 - h_{298})_i\right]_{\text{reactants}} = \left[\Sigma_i N_i (\Delta \bar{h}_f^\circ)_i\right]_{\text{products}} + \left[\Sigma_i n_i (\bar{h}_2 - h_{298})_i\right]_{\text{products}} \tag{5.29}
$$

We can rewrite this as...

$$
{}_1q_2 = \left[\Sigma_i N_i (\bar{h}_2 - h_{298})_i\right]_{\text{products}} - \left[\Sigma_i N_i (\bar{h}_1 - h_{298})_i\right]_{\text{reactants}} + \Delta \bar{h}_r^\circ \tag{5.30}
$$

(5.30) = 0 if we have an adiabatic combustion. We use this to find the maximum temperature possible.

5.5.4 Adiabatic Flame Temperature

Here we use the Energy Equation for Chemical Reactions (section 3 above), to predict the maximum temperature the products of a reaction might have. It's a maximum temperature because we are assuming adiabatic combustion, that is, no heat is lost from the combustor. All the bond energy and temperature energy of the reactants goes into the bond energy and temperature energy of the products.

So, ${}_1q_2$ in (5.30) :

$$\left[\Sigma_i N_i(\bar{h}_2 - h_{298})_i\right]_{\text{products}} = \left[\Sigma_i N_i(\bar{h}_1 - h_{298})_i\right]_{\text{reactants}} - \Delta\bar{h}_r^\circ$$

$(\bar{h}_2 - h_{298})$ can be found in handout 1h for different species.

Enthalpy changes to raise a particular species from 298K (reference temp) to another temp T, is given in handout 1h. We could use $(\bar{h} - h_{298}) = C_P(T - 298)$ but this is only approximate if C_p is constant while the tables do not.

Example: Given the chemical reaction:

$$H_2 + \frac{1}{2}O_2 \longrightarrow H_2O(g)$$

What is the max temp possible for this reaction?

1. Compute $\Delta\bar{h}_r^\circ$

$$\Delta\bar{h}_r^\circ = [-57795] - [1(0) + \frac{1}{2}(0)] = -57795 \qquad [cal/mol]$$

2. Use table for $h - h_{298}$ at 600K

$$(h - h_{298})_{O_2} = 2210 \qquad [cal/mol]$$
$$(h - h_{298})_{H_2} = 2106 \qquad [cal/mol]$$

3. Plug into equation (5.34)

$$\begin{aligned}(1)(h - h_{298}) &= (1)(h - h_{298})_{H_2} + \frac{1}{2}(1)(h - h_{298}) - \Delta\bar{h}_r^\circ \\ &= 2106 + \frac{1}{2}(2210) - (-57795) \\ &= 61006\,\frac{cal}{mol}\end{aligned}$$

4. Now do a reverse look up in the table to find the temperature. Find an $h - h_{298}$ which we find corresponds to $T_2 = 5300\,K$

What is wrong with this result?

We have ignored the fact that at this temperature, the water will be dissociated. We don't just have water as a reactant. We will also have H, HO, H_2. We definitely have dissociation since dissociation becomes important at around 2000K.

We need to modify this as:

$$H_2 + \frac{1}{2}O \longrightarrow \mathscr{N}_{H_2O}\, H_2O + \mathscr{N}_{O_2}\, O_2 + \mathscr{N}_{H_2}\, H_2 + \mathscr{N}_O\, O + \mathscr{N}_H\, H + \mathscr{N}_{OH}\, OH$$

But to find the dissociation we need temperature. Therefore, we guess a temperature, then solve for the unknown equilibrium composition. Then you go to the adiabatic flame temperature calculation. You come up with a new temperature then you repeat. You iterate until it converges to the actual temperature of the process.

Section 2: Rockets Basics

AE435

Spring 2018

1 Rockets Basics

Rockets carry their propellant with them. Useful in atmosphere and space flight

Contents

1.1 Definitions and Fundamentals

1.1.1 Definitions

Total Impulse - I_t is the thrust force, F, integrated over the burn time, t_b

$$I_t = \int_0^{t_b} F\,\mathrm{d}t \qquad [N-s] \tag{6.1}$$

F can be a function of time, t. Note this It has units of $N \cdot sec$. It is related (proportional to) the total energy released by all propellant in the propulsion system.

Specific impulse - I_{sp}, total impulse per unit weight of propellant. Important figure of merit for rocket performance. Similar to "miles-per-gallon" rating in a car.

$$I_{SP} = \frac{\int_0^{t_b} F\,\mathrm{d}t}{g\int_0^{t_b} \dot{m}\,\mathrm{d}t} \qquad [sec] \tag{6.2}$$

where g is the gravitational constant and we always use $g = 9.81 m/s^2$ and $\dot{m}$ is the mass flow rate.

If constant thrust and flow rate, then:

$$I_{SP} = \frac{I_t}{m_p\,g} = \frac{F\,t_b}{g\,m_p} = \frac{F}{g\,\dot{m}} = \frac{F}{\dot{W}} \tag{6.3}$$

Where $\dot{W}$ is the weight flow rate.

Effective exhaust velocity - is the average equivalent velocity at which propellant is ejected from the vehicle.

$$c = I_{SP}\,g = \frac{F}{\dot{m}} \qquad [m/s] \tag{6.4}$$

Mass Ratio - is final mass of vehicle (all propellant consumed and ejected) divided by initial mass (before rocket operation)

$$MR = \frac{m_f}{m_o} \tag{6.5}$$

Propellant Mass Fraction - is fraction of propellant in initial mass.

$$\xi = \frac{m_p}{m_o} \tag{6.6}$$

Where $m_o = m_f + m_p$ so...

$$1 = \frac{m+p}{m_o} + \frac{m_f}{m_o} = \xi + MR$$

$$\xi = 1 - MR \tag{6.7}$$

Thrust-to-weight ratio - it expresses acceleration that rocket is capable of giving to its own loaded propulsion system mass.

$$\frac{F}{W_o}$$

Which could be initial or instantaneous.

Example: Rocket has following characteristics:

$$M_o = 200\,kg$$
$$M_f = 130\,kg$$
$$Payload = 110\,kg$$
$$OperationTime = 3.0\,sec$$
$$Avg.I_{sp} = 240\,sec$$

So this rocket has a mass ratio of...

$$MR = \frac{M_f}{M_o} = \frac{130}{200} = 0.65 \quad \text{mass ratio of the vehical}$$

$$MR = \frac{M_f}{M_o} = \frac{130-1100}{200-110} = 0.222 \quad \text{mass ratio of the propulsion system}$$

$$\xi = 1 - MR = 1 - 0.65 = 0.35 \quad \text{Vehical Propellant Mass Fraction}$$

$$m_p = 200 - 130 = 70kg$$

$$\dot{m} = \frac{70kg}{3.0sec} = 23.3\ [kg/s]$$

$$F = I_{SP}\,\dot{w} = 240 * 23.3 * 9.81 = 54800\,N\text{thrust}$$

Therefore

$$\left.\frac{F}{W}\right|_{\text{initial}} = \frac{54800}{200 * 9.81} = 28\,g's$$

$$\left.\frac{F}{W}\right|_{\text{final}} = \frac{54800}{130 * 9.81} = 43\,g's$$

1.1.2 Thrust

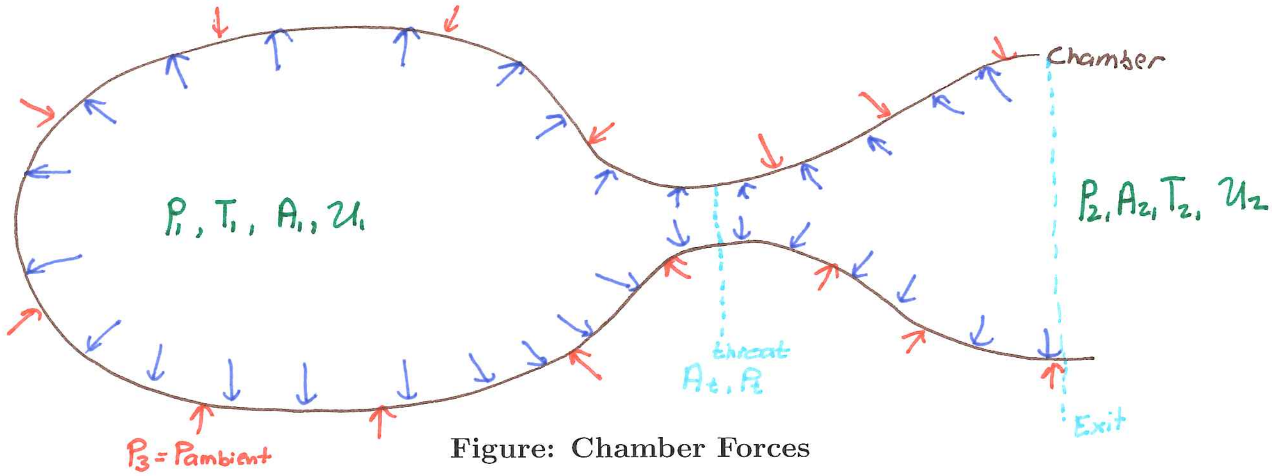


Figure: Chamber Forces

Draw the control volume, apply conservation equations and momentum equation.

$$\text{Thrust} = F = \dot{m}\, u_2 + (P_2 - P_3)\, A_2 \tag{6.8}$$

This is the same as equation 3.13 however we have no inlet velocity, $u_i = 0$ because rockets don't have inlets.

- First term is momentum thrust $= \dot{m}\, u_2$
- 2nd term is pressure thrust $= (P_2 - P_3)\, A_2$
- In vacuum, $P_3 = 0$

In atmosphere flight $P_3 = P_a$, will change with altitude, so pressure thrust also changes. Thrust and I_{SP} actually increase with altitude. Pressure thrust is not negligible. Around 20% of it can be for thrust. When you have enough information to calculate it, you should calculate it.

1.1.3 Exhaust Velocity

From (6.4) and (6.8), $c = \frac{F}{\dot{m}}$ can be written as:

$$F \approx \dot{m}\, u_2 \tag{6.9}$$

We have neglected the pressure thrust. This implies that $c = u_2$. More rigorously,

$$c = u_2 + \frac{(P_2 - P_3)\, A_2}{\dot{m}} \tag{6.10}$$

Characteristic velocity: Purely a function of the combustion chamber properties like the mass flow rate.

$$c^* = \frac{P_1\, A_t}{\dot{m}} \tag{6.11}$$

Related to combustion efficiency, independent of nozzle, used to compare performance of chemical propulsion system design and propellant. More of a comparison factor. Use this to compare it with other thruster or propellants.

I_{SP}, and c, are functions of nozzle geometry, such as area ratio $\frac{A_2}{A_t}$. They are also functions of back pressure (ambient pressure), hence many rockets typically have a vacuum ISP and sea level ISP. (Vac ISP is higher because that sea level despite larger expansion angle due to no atmospheric pressure to push against)

1.1.4 Energy & Efficiency

Jet power, P_{jet}, is the time rate of expenditure of ejected matter Kinetic Energy.

$$P_{\text{jet}} = \frac{1}{2}\dot{m}\,c^2 = \frac{1}{2}\dot{w}\,g\,I_{SP}^2 = \frac{1}{2}F\,g\,I_{SP} = \frac{1}{2}F\,c \tag{6.12}$$

If you want more thrust, I_{SP} must go down. If you want more I_{SP} or effective exhaust velocity, thrust will go down.

Internal efficiency: how effectively the energy in the propellant gets converted into thrust.

$$\eta_{int} = \frac{\text{KE in Jet}}{\text{available chem power}} = \frac{\frac{1}{2}\dot{m}c^2}{\eta_{comb}\,P_{chem}} \tag{6.13}$$

$$\eta_{\text{combustion}} = \frac{\text{Internal Energy of Gas}}{P_{\text{chem}}}$$

where P_{chem} is the power input into the combustion chamber in the form of chemical reactions. It tells us the fraction of bond energy that gets burnt up and used as thrust

Power transmitted to the vehicle:

$$P_{veh} = F\,u \tag{6.14}$$

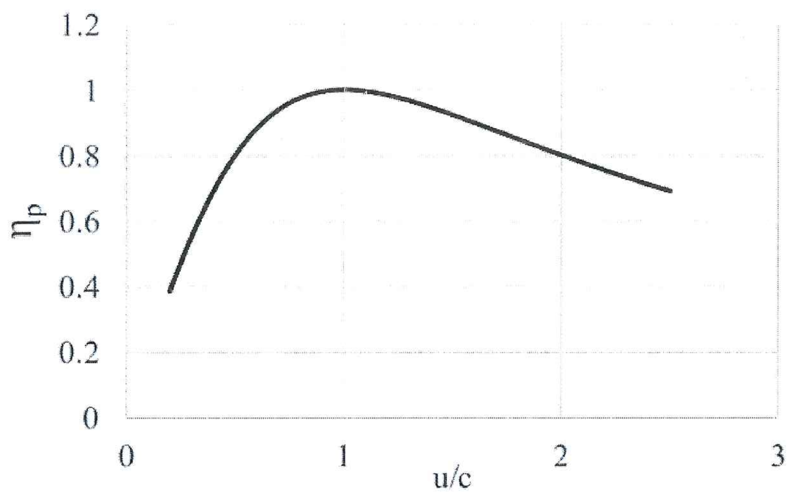
where u is the vehicle velocity.

Propulsive efficiency

$$\eta_P = \frac{\text{Vehicle Power}}{\text{Vehicle Power} + \text{Residuual KE Jet Power}}$$

$$= \frac{F\,u}{F\,u + \frac{1}{2}(\dot{w}/g)\,(c-u)^2} \tag{6.15}$$

$$= \frac{2\,(u/c)}{1+(u/c)^2}$$



Max efficiency when $u = c$, exhaust gases stand still in free space.

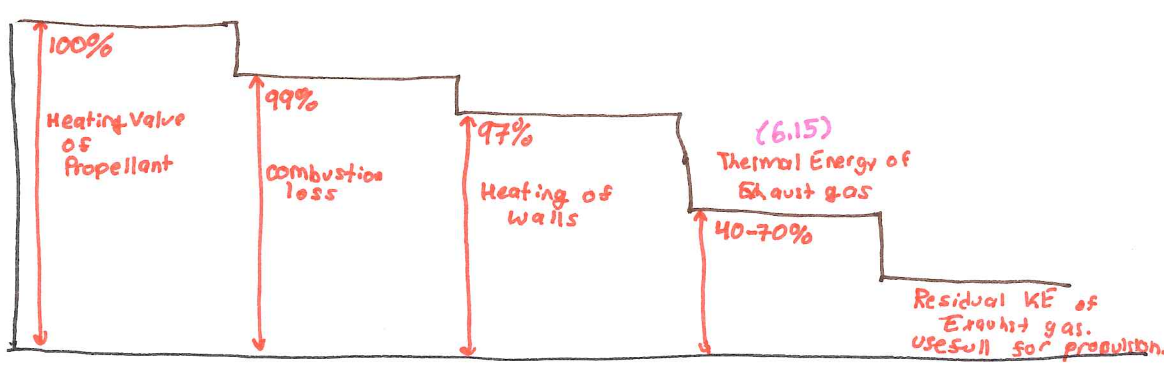


Figure: Efficiency Losses

However, minimizing mass is often more important than max efficiency. And c or I_{SP} become more important since they are a measure of this mass economy.

We typically talk about rockets by how efficiently they use propellant mass. This is determined by your effective exhaust velocity and specific impulse. How efficiently does the thruster use the propellant mass in order to create impulse.

1.1.5 Typical Performance Parameters

TABLE 2-1. Ranges of Typical Performance Parameters for Various Rocket Propulsion Systems

Engine Type	Specific Impulse[a] (sec)	Maximum Temperature (°C)	Thrust-to-Weight Ratio[b]	Propulsion Duration	Specific Power[c] (kW/kg)	Typical Working Fluid	Status of Technology
Chemical—solid or liquid bipropellant	200–410	2500–4100	10^{-2}–100	Seconds to a few minutes	10^{-1}–10^{3}	Liquid or solid propellants	Flight proven
Liquid monopropellant	180–223	600–800	10^{-1}–10^{-2}	Seconds to minutes	0.02–200	N_2H_4	Flight proven
Nuclear fission	500–860	2700	10^{-2}–30	Same	10^{-1}–10^{3}	H_2	Development was stopped
Resistojet	150–300	2900	10^{-2}–10^{-4}	Days	10^{-3}–10^{-1}	H_2, N_2H_4	Flight proven
Arc heating—electrothermal	280–1200	20,000	10^{-4}–10^{-2}	Days	10^{-3}–1	N_2H_4,H_2,NH_3	Flight proven
Electromagnetic including Pulsed Plasma (PP)	700–2500	—	10^{-6}–10^{-4}	Weeks	10^{-3}–1	H_2 Solid for PP	Flight proven
Hall effect	1000–1700	—	10^{-4}	Weeks	10^{-1}–5×10^{-1}	Xe	Flight proven
Ion—electrostatic	1200–5000	—	10^{-6}–10^{-4}	Months	10^{-3}–1	Xe	Several have flown
Solar heating	400–700	1300	10^{-3}–10^{-2}	Days	10^{-2}–1	H_2	In development

[a] At $p_1 = 1000$ psia and optimum gas expansion at sea level $Ip_2 = p_3 = 14.7$ psia).
[b] Ratio of thrust force to full propulsion system sea level weight (with propellants, but without payload).
[c] Kinetic power per unit exhaust mass flow.

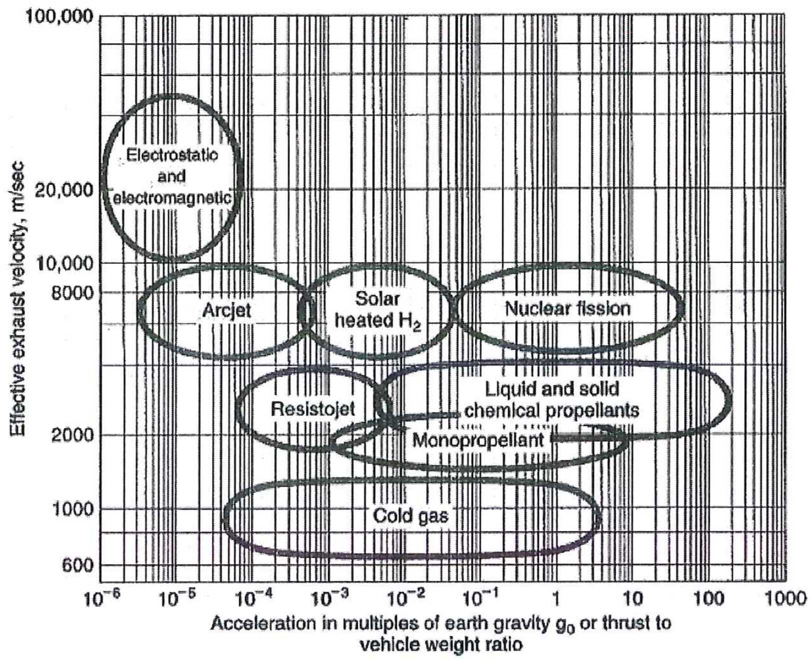


FIGURE 2–5. Exhaust velocities as a function of typical vehicle accelerations. Regions indicate approximate performance values for different types of propulsion systems. The mass of the vehicle includes the propulsion system, but the payload is assumed to be zero.

1.2 Rocket Performance in Gravity Free Space

- Most basic way to compare engines
- Assume no drag, no aerodynamic loads, no gravity (i.e., 3 light years from Earth)
- Thrust direction is flight direction, straight-line acceleration

Newton's 2nd Law:

$$F = m\frac{\mathrm{d}v}{\mathrm{d}t} = \dot{m}\,c = -\frac{\partial m}{\partial t}c \tag{6.16}$$

The last term is negative because we have a negative mass change for the vehicle

$$\Delta v = -\int_{m_o}^{m_f} c\frac{\mathrm{d}m}{m} \tag{6.17}$$

Where

$$\begin{aligned} m_o &= \text{Initial Mass} \\ m_f &= \text{Final Mass} \\ m_p &= \text{Propellant Mass} \\ c &= \text{Effective Exhaust Velocity} \end{aligned}$$

then

Rocket Equation in Gravity Free Space

$$\frac{\Delta v}{c} = \ln\left(\frac{m_o}{m_f}\right) = -\ln(1-\xi) = \ln\left(\frac{1}{MR}\right) \tag{6.18}$$

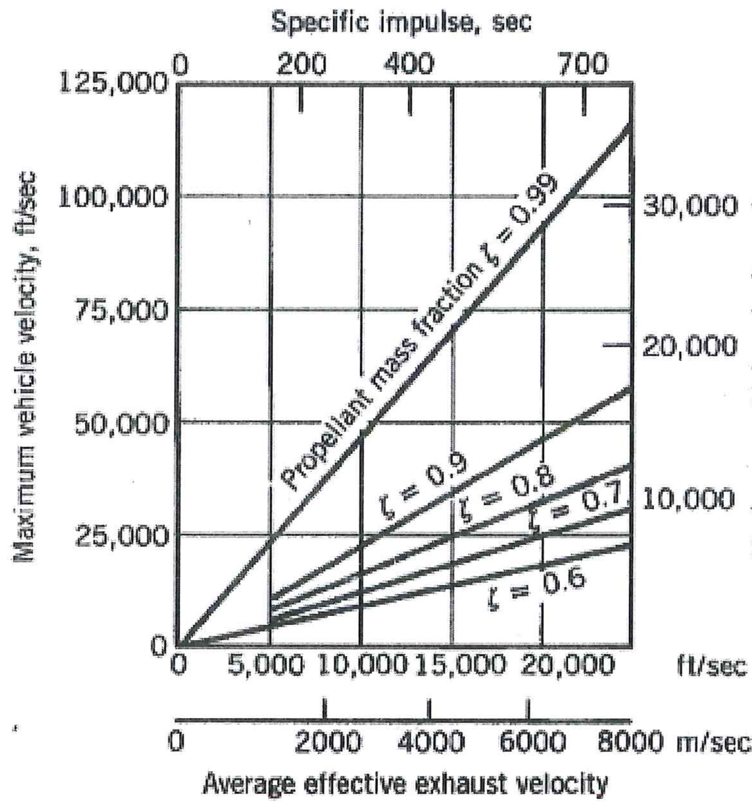
Example: Consider a rocket with I_{SP}=400s, 1/MR = 10, (90% of m_o is propellant)

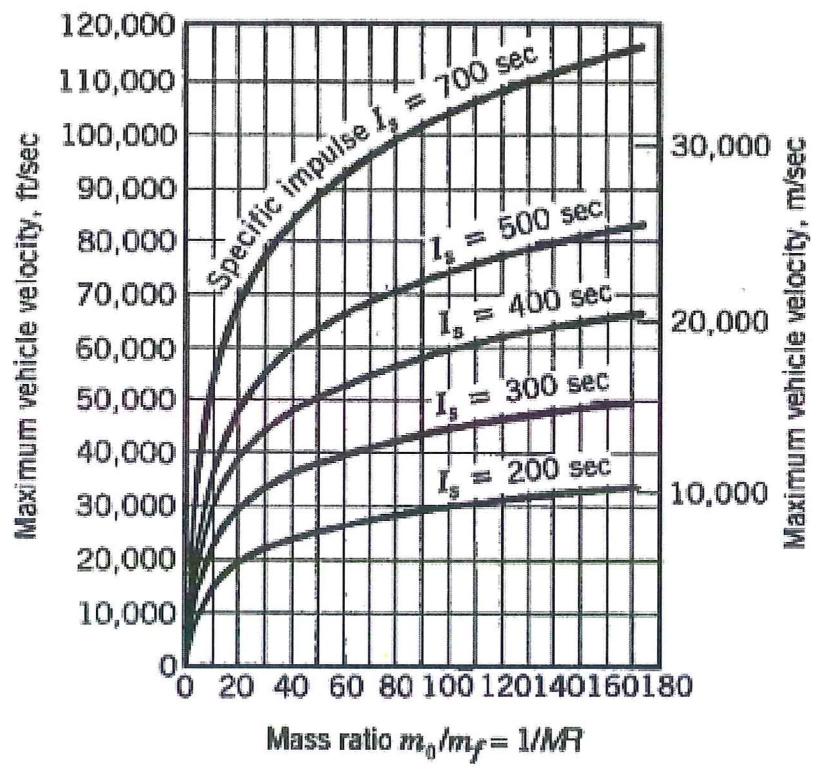
$$\Delta v = c\ln\left(\frac{1}{MR}\right) = I_{SP}\,g\,\ln\left(\frac{1}{MR}\right) = 400\,(9.81)\,\ln(10)$$

$$= 9040\,[m/s]$$

General Trends of the gravity-free-space rocket equation: (6.18)

- An increase in c or I_{SP} enables a larger Δv
- An increase in propellant mass fraction (more propellant onboard) enables larger Δv
- A smaller mass ratio ($MR = m_f/m_o$) means more propellant, so enables larger Δv


Specific impulse, sec
0
200
400
700
125,000
100,000
75,000
50,000
25,000
0
Maximum vehicle velocity, ft/sec
30,000
20,000
10,000
Maximum vehicle velocity, m/sec
Propellant mass fraction ζ = 0.99
ζ = 0.9
ζ = 0.8
ζ = 0.7
ζ = 0.6
0 5,000 10,000 15,000 20,000
ft/sec
0 2000 4000 6000 8000 m/sec
Average effective exhaust velocity


120,000
110,000
100,000
90,000
80,000
70,000
60,000
50,000
40,000
30,000
20,000
10,000
0
Maximum vehicle velocity, ft/sec
30,000
20,000
10,000
Maximum vehicle velocity, m/sec
Specific impulse I_s = 700 sec
I_s = 500 sec
I_s = 400 sec
I_s = 300 sec
I_s = 200 sec
0 20 40 60 80 100 120 140 160 180
Mass ratio $m_0/m_f = 1/MR$

What we will do now is rewrite the rocket equation in different forms to analyze the effect of thrust-to-weight ratio, payload ratio, and structural efficiency on rocket performance.

1.2.1 Thrust-to-Weight Ratio (F/W_o)

Rocket Eqn:

$$1 - \frac{m_p}{m_o} = \exp\left(\frac{-\Delta v}{c}\right) \tag{6.19}$$

What does rocket equation tell us about F/Wo in terms of burn time, t_b, I_{SP} and Δv?

We will assume constant mass flow rate, and a constant effective exauhst velocity such that...

$$\dot{m} = \text{const} \quad \rightarrow \quad m_p = \dot{m}\, t_b$$
$$c = \text{const} \quad \rightarrow \quad c = I_{SP}\, g$$

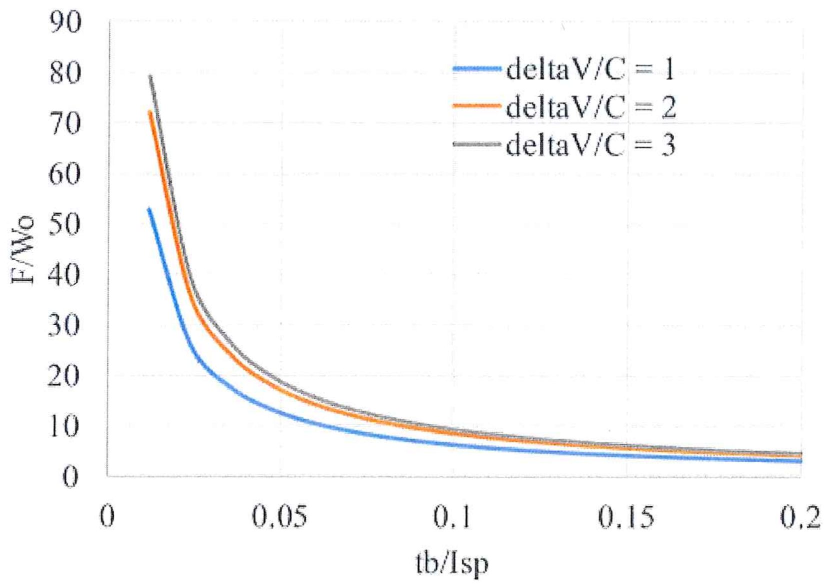
$$F = \dot{m}\, c \quad \rightarrow \quad m_p = \frac{F\, t_b}{c} \tag{6.20}$$

$$\frac{F\, t_b}{m_o\, c} = 1 - \exp\left(-\frac{\Delta v}{c}\right) \tag{6.21}$$

Then

Thrust-to-Weight Ratio in Gravity Free Space

$$\frac{F}{w_o} = \frac{I_{SP}}{t_b}\left(1 - \exp\left(-\frac{\Delta v}{c}\right)\right) \tag{6.22}$$



- Starting burn with higher initial acceleration (Larger F/W_o) , for a burn time t_b yields more ΔV
- Or, more time burning yields more Δv

For Example: Consider a rocket with $I_{SP} = 400$s, $\frac{1}{MR} =$ which impl;ies that $\Delta v \approx 9\,[km/s]$

For an initial $F/W_o = 1$, then (6.22) tells us

$$t_b = 360\,[\text{sec}] = 6\,[\text{min}]$$

This rocket needs to burn for 6 minutes in order to achieve a 9 km/s Δv.

What if we reduced the initial thrust to weight ratio.

Lets say now $F/W_o = 1/4$, now $t_b = 24$ min

If we let $r = F/W_o$, and using (6.18) and letting $l = 1/MR$, then (6.22) becomes:

$$r = \frac{I_{SP}}{t_b}\left(1 - \frac{1}{l}\right) \tag{6.23}$$

or

Dimensionless Burn Time

$$\frac{t_b}{I_{SP}} = \frac{1}{r}\left(1 - \frac{1}{l}\right) \tag{6.24}$$

1.2.2 Payload Ratio and Structural Efficiency

$$\begin{aligned} m_L &= \text{Payload Mass} \\ m_p &= \text{Propellant Mass} \\ m_{st} &= \text{Structure Mass (Vehicle Dry Mass - Payload Mass)} \end{aligned}$$

$$m_o = m_L + m_p + m_{st}$$

$$m_f = m_L + m_{st}$$

We define...

Structural Efficiency:

$$\varepsilon = \frac{m_{st}}{m_o - m_L} = \frac{m_{st}}{m_p + m_{st}} \tag{6.25}$$

Payload Ratio:

$$\lambda \equiv \frac{m_L}{m_o} \tag{6.26}$$

Rocket Equation:

$$\begin{aligned} 1 - \frac{m_p}{m_o} &= \exp\left(-\frac{\Delta v}{c}\right) \\ &= \frac{m_L + m_{st}}{m_o} \\ &= \frac{m_{st}}{m_o} + \lambda \end{aligned} \tag{6.27}$$

Now...

$$\frac{m_{st}}{m_o} = (1 - \lambda)\,\varepsilon \tag{6.28}$$

so, from (6.27) and (6.28),

$$(1 - \lambda)\varepsilon + \lambda = \exp\left(-\frac{\Delta v}{c}\right) \tag{6.29}$$

Payload Ratio ito Structural Efficiency and Δv

$$\lambda = \frac{m_L}{m_o} = \frac{\exp\left(-\frac{\Delta v}{c}\right) - \varepsilon}{1 - \varepsilon} \tag{6.30}$$

From (6.30) we see that as $\lambda \to 0$, the payload goes away.

Then, this reduces to...

$$\frac{\Delta V}{c} = -\ln\varepsilon \tag{6.31}$$

This tells us the maximum ΔV possible for this rocket. This tells us that the structure mass limits the total ΔV that can be achieved. This means that we want a structural efficiency ϵ to be as low as possible. Typically we see $\epsilon \approx 0.1$

1.3 Vertical Flight Constant Thrust & I_{SP}

Previously, we were operating in gravity-free space. Now we add g.

- Constant gravity, g, and NO aerodynamic forces

Eqn. of Motion:

always g of Earth

$$\frac{dv}{dt} = \frac{F}{m} - g = \frac{g\, I_{SP}\, \dot{m}}{m} - g \tag{6.32}$$

g local

What assumption was made between (6.32) and (6.33)?

We made made the assumption that the rocket is always moving upwards, gravity is constant and does not change with altitude difference.

Note that the g when multiplied by I_{SP} is always earth gravity and the other two g's are local gravity. (6.33) makes the assumption that we are on earth and all g's are local gravity of earth.

$$\frac{1}{g}\frac{dv}{dt} = \frac{I_{SP}\, \dot{m}}{m} - 1 = \frac{a}{g} \tag{6.33}$$

1.3.1 Acceleration

@ $t = 0$, $m = m_o$, and $v = 0$, a_o = initial acceleration, then

$$\frac{a_o}{g} = \frac{I_{SP}\, \dot{m}}{m_o} - 1 \tag{6.34}$$

@ $t = t_b$, $m = m_f = m_o/l$, and a= $a_{\max}$, then

$$\frac{a_{max}}{g} = \frac{I_{SP}\, l\, \dot{m}}{m_o} - 1 = \frac{I_{SP}\, l\, m_p}{t_b\, m_o} - 1 \tag{6.35}$$

$$\frac{a_{max}}{g} = n_{max} = \frac{I_{SP}}{t_b}(l-1) - 1 \tag{6.36}$$

acceleration in g's

Assuming constant thrust and mass flow rate.

The dimensionless burn time is then:

$$\frac{t_b}{I_{SP}} = \frac{l-1}{n_{max}+1} \tag{6.37}$$

So, t_b/I_{SP} decreases when:

- $l = \frac{m_o}{m_f}$ decreases (as m_f increases, less mp available, t_b decreases, can't burn as long)
- n_{max} increases (n_{max} increases means more acceleration, means increased mass flow rate, so t_b decreases cause expending propellant faster)

1.3.2 Loading

Max g-load on Payload

$$n_{\max} + 1$$

$n_{\max} + 1$ = max g-load on payload, includes g-load due to rocket acceleration under its thrust and the ambient gravity g (the +1)

Max loading dictated by payload (typically <~ 6-7g's)

Rearranging (6.37):

Loading Factor

$$\underbrace{n_{\max} + 1}_{\gamma} = \frac{l-1}{\frac{t_b}{I_{SP}}} = r\,l \tag{6.38}$$

$\gamma \frac{t_b}{I_{SP}} = \frac{(l-1) I_{SP}}{\gamma}$

Where $r = \frac{F}{W_o}$

1.3.3 Burn Time

Again from equation of motion:

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = F - m\,g$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{F}{m} - g = \frac{\dot{m}\,c}{m} - g = -\frac{c}{m}\frac{\mathrm{d}m}{\mathrm{d}t} - g \tag{6.39}$$

Integrate and we find that

$$\Delta v = c\,\ln\left(\frac{m_o}{m_f}\right) - t_b\,g \tag{6.40}$$

or

$$\frac{\Delta v}{c} = \ln(l) - \frac{t_b}{I_{SP}} \tag{6.41}$$

This tells us that to get max deltaV, need short t_b (impulsive burn) and small mass ration ($m_f \ll m_o$). Rocket needs to be all propellant, unless Isp large. Hence research into high I_{SP}!

Rocket Eqn, written in terms of:

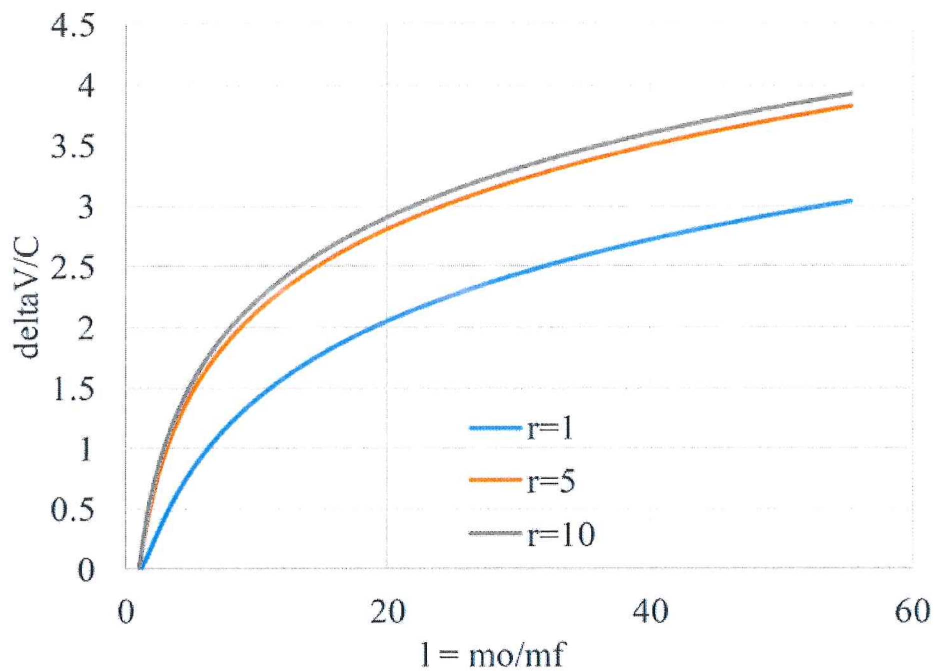
Compare:

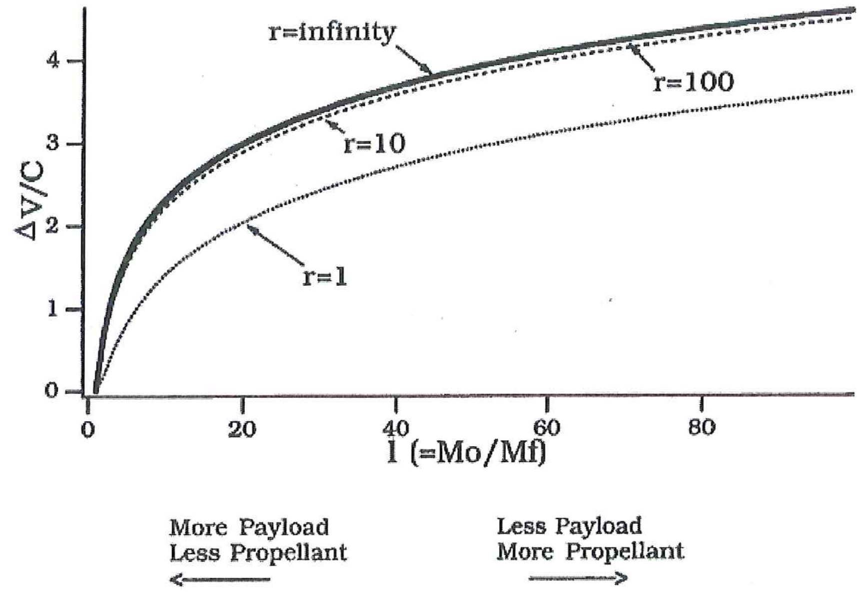
A) $\frac{\Delta v}{c}$, l , $\frac{t_b}{I_{SP}}$

$$\frac{\Delta v}{c} = \ln(l) - \frac{t_b}{I_{SP}} \tag{6.42}$$

B) $\frac{\Delta v}{c}$, l , r

$$\frac{\Delta v}{c} = \ln(l) - \frac{1}{r}\left(1 - \frac{1}{l}\right) \tag{6.43}$$

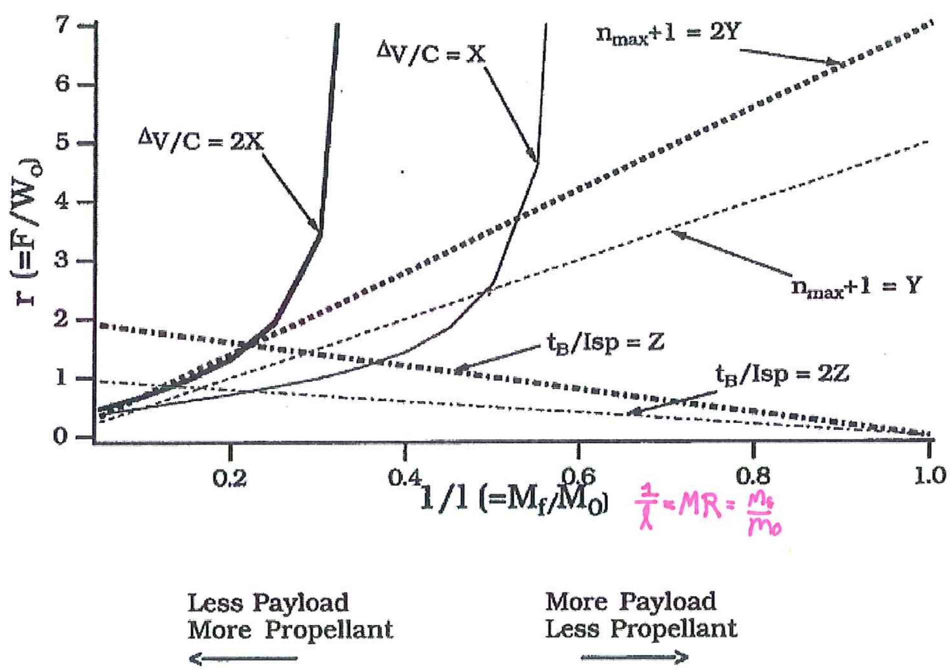




C) $\frac{\Delta v}{c}$, l , $n_{\max}$

$$\frac{\Delta v}{c} = \ln(l) - \frac{l-1}{n_{max}+1} \tag{6.44}$$

All these equations/results are illustrated in this figure. From this figure you can pick off, for a given mass ratio and initial thrust-to-weight ratio, the burn time, max loading, and deltaV for the rocket.



For a fixed initial thrust-to-weight ratio (acceleration) and I_{SP}:

- Higher Δv requires smaller mass ratio (rocket needs to be more propellant, less payload)
- Burn time increases for smaller mass ratios (because the rocket is more propellant, less payload)
- Max loading increases for smaller mass ratio (because it's constant thrust, so the max loading is at the end of the burn, and smaller mass ratio means smaller final mass, so same force/thrust is being applied to smaller mass, hence higher loading)

Conversely, for fixed mass ratio and I_{SP}:

- Higher Δv requires larger initial thrust-to-weight ratio (need more acceleration to get to that higher Δv)
- Burn time decreases for larger thrust-to-weight ratio (more thrust, more acceleration means higher mass flow rate, burn through your propellant faster)
- Max loading increases for increasing initial thrust-to-weight ratio ($F = m\,a$, the final mass is now fixed, so as thrust/F increases, so does acceleration, that is, the max loading)

SEE HANDOUT 2F also

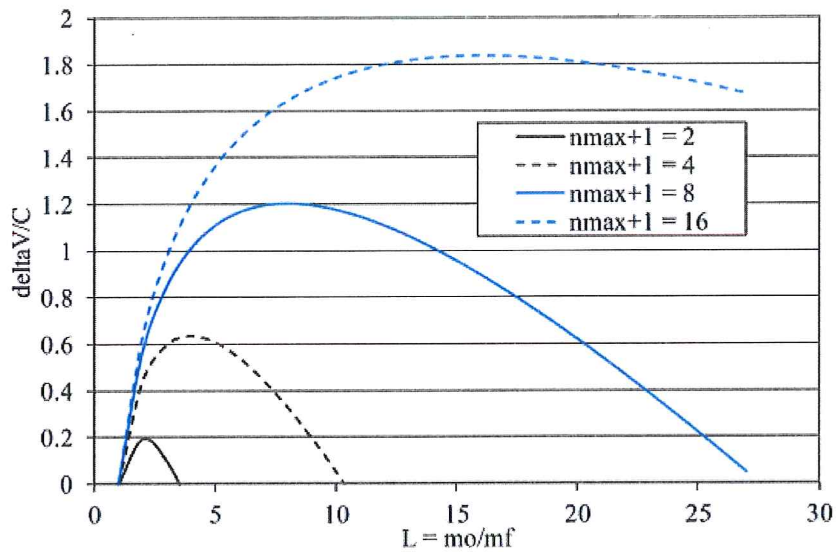
1.4 Constant Thrust but Acceleration Limited Vertical Boost

Now we limit $n_{max}+1$. What is the max Δv possible for given t_b, g-load, and mass ratio? In other words, is there an optimium l with fixed n_max that maximizes Δv?

$$\frac{\Delta v}{c} = \ln(l) - \frac{l-1}{n_{max}+1}$$

Let the optimum mass ration, $l = l*$, how to find the maximum of this function? Take derivative and set equal to zero.

$$\frac{d}{dl}\left(\frac{\Delta v}{c}\right) = \frac{1}{l} - \frac{1}{n_{max}+1} = 0 \rightarrow l = l^* = n_{max}+1 \tag{6.51}$$



From (6.38), $n_{max}+1 = rl$. At optimum $l = l^*$, this implies: $r = 1$. So, at liftoff want $r = 1 = F/W_o$

In cases where

- $n_{max}+1$ not limited, choose highest r.
- $n_{max}+1$ is limited, set $r = 1$.

Then, from (6.37) with $l = l^*$,

$$\left.\frac{\Delta v}{c}\right|_{max} = \ln(n_{max}+1) - \frac{n_{max}}{n_{max}+1} \tag{6.52}$$

Note, this is for a constant thrust. Eqn. (6.37) was for constant thrust.

1.5 Constant Acceleration Vertical Boost

$$\frac{\mathrm{d}}{\mathrm{d}t} = \text{const} = \frac{F}{m} - g = a \tag{6.53}$$

$$I_{SP} = \text{const} = \frac{c}{g}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\dot{m}\,c}{m} - g = a = \text{const} \tag{6.54}$$

Since $m = m(t)$ mass of the vehicle is changing with time.

$$\dot{m} = \dot{m}(t) \qquad \text{and} \qquad F = F(t)$$

Let our constant $a = n_{max} g$ Then:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = n_{max}\,g \rightarrow \Delta v = n_{max}\,g\,t_b \tag{6.55}$$

From (6.40) (we did not require constant thrust for this equation)

$$\Delta v = c\,\ln(l) - g\,t_b \tag{6.56}$$

Combine (6.40) & (6.56) to get

Delta V of Constant Acceleration Rocket

$$\frac{\Delta v}{c} = \frac{n_{max}}{n_{max} + 1}\ln(l) \tag{6.57}$$

This is DeltaV of Constant Acceleration Rocket

We should expect constant acceleration case to give higher Δv for given mass ratio (MR or $1/MR = l$). Because rocket is at the max loading/max acceleration all the time, not just at burnout, t_b.

Example: Payload with g-limit 6 g's

It's acceleration limited, so:

For constant thrust trajectory, set $n_{max} + 1 = l^* = 6g's$, then with (6.52),

$$\left.\frac{\Delta v}{c}\right|_{max} = \ln(6) - \frac{5}{6} = 0.96$$

For constant acceleration trajectory (6.57) with

$$\frac{\Delta v}{c} = 0.96 = \frac{5}{6}\ln(l) \rightarrow l = 3.2$$

	F Constant	Acceleration Constant
$\frac{\Delta v}{c}$	0.96	0.96
$l = \frac{1}{MR} = \frac{m_o}{m_f}$	6	3.2
t_b for I_{SP} =200sec	208	51

Section 2: Rockets Basics

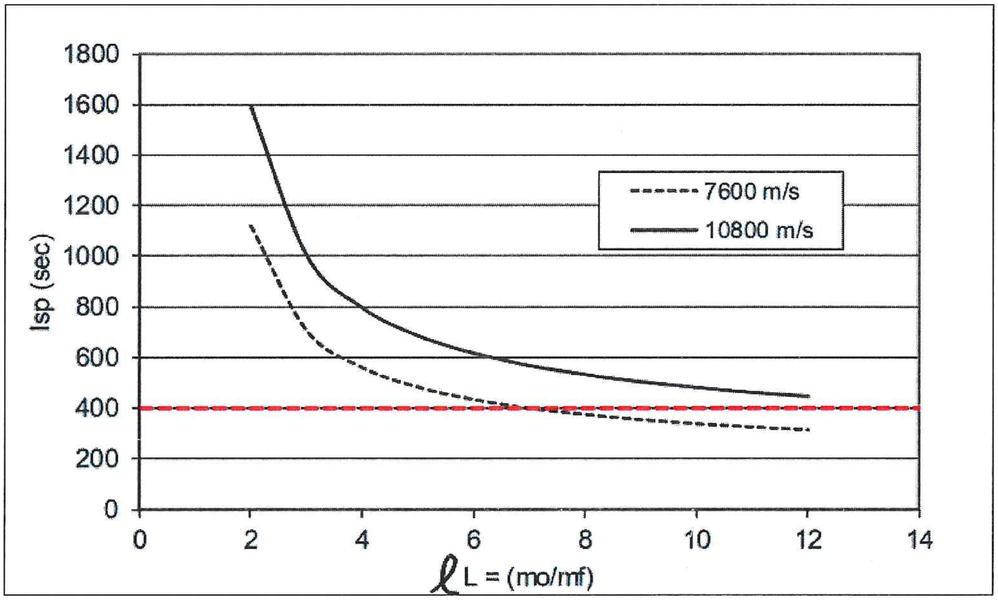
AE435

Spring 2018

2 Multi-stage Rockets

Contents

2.1 Introduction



SINGLE STAGE TO ORBIT

Single stage to orbit with SSME (425 sec, state-of-the-art) Isp and mass ratio for a deltaV - Earth to LEO (7600m/s) Factor in Drag, Vertical to Horizontal velocity, launch location (10km/s)

For Space Shuttle:

- total delta-V = 9347 m/s
- actual orbit velocity is only = 7790 m/s
- Gravity loss = 1080 m/s
- Drag loss = 120 m/s
- Vertical-Horizontal = 360 m/s

$L = 7$ for$7600m/s \rightarrow$ 86%propellant, $\sim$ 14% mass is final mass (which includes structure!, which is probably all the mass, no room for payload)

For actual deltaV (10km/s), nothing left, $L > 12, > 92\%$ propellant, can't do it?

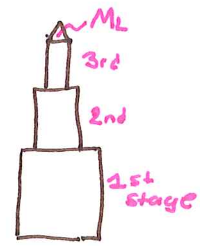
So, use multiple stages. Leave used empty structure behind. Discard spent structure, engines, tankage.

Analysis similar to single-stage since payload for any stage is mass of all subsequent stages.

2.2 Definitions

Consider the ith stage

$$m_{oi} = \text{total initial mass of ith stage}$$
$$m_{fi} = \text{final mass of ith stage}$$
$$m_{st,i} = \text{structure mass of ith stage}$$
$$m_{pi} = \text{propellant mass of ith stage}$$
$$m_L = \text{payload mass of last stage}$$



Then from our prior analysis:

Payload ratio:

$$\lambda_i = \frac{M_{o,(i+1)}}{M_{o,i} - M_{o,(i+1)}} \tag{7.1}$$

Mass ratio:

$$\mathscr{MR}_i = \frac{M_{f,i}}{M_{o,i}} = \frac{1}{\mathscr{R}_i} \tag{7.2}$$

Structural efficiency:

$$\varepsilon_i = \frac{M_{st,i}}{M_{o,i} - M_{o,(i+1)}} \tag{7.3}$$

$$= \frac{M_{f,i} - M_{o,(i+1)}}{M_{o,i} - M_{o,(i+1)}} \tag{7.4}$$

Then

$$\mathscr{MR}_i = \frac{\varepsilon_i + \lambda_i}{1 + \lambda_i} = \frac{1}{\mathscr{R}_i} \tag{7.5}$$

2.3 Vertical Boost

Rocket Eqn for the ith stage (see (6.40))

$$(\Delta V_b)_i = c_i \ln\left(\frac{1}{\mathscr{MR}_i}\right) - g\, t_{b,i} \tag{7.6}$$

$$= I_{SP_i}\, g \ln\left(\frac{1}{\mathscr{MR}_i}\right) - g\, t_{b,i} \tag{7.7}$$

For an N-stage vehicle then:

$$(\Delta V_b)_N = \sum_{i=1}^{N} c_i \ln\left(\frac{1}{\mathscr{MR}_i}\right) - g\, T_b \tag{7.8}$$

Where T_b is the total burn time.

Assume ε constant stage-to-stage: same design practices and materials

Assume c_i constant stage-to-stage: each stage same I_{sp}.

Then

$$(\Delta V_b)_N = c \sum_{i=1}^{N} \ln\left(\frac{1+\lambda_i}{\varepsilon+\lambda_i}\right) - g\, T_b \tag{7.9}$$

2.4 Optimum Payload Ratio for Maximum Delta V

For given payload m_L, and total vehicle mass m_o, how do we determine λ_i the payload ratio of each stage, to maximize ΔV ??

Assume each stage has same I_{sp} and same structural efficiency, ε. From (7.1)

$$\frac{m_{o,(i+1)}}{m_{oi}} = \frac{\lambda_i}{1+\lambda_i} \tag{7.10}$$

$$\frac{(M_o)_N}{(M_o)_1} = \prod_{i=1}^{N} \left(\frac{\lambda_i}{1+\lambda_i}\right) \tag{7.11}$$

Then

$$\ln\left(\frac{(M_o)_N}{(M_o)_1}\right) = \ln\left(\prod_{i=1}^{N}\left(\frac{\lambda_i}{1+\lambda_i}\right)\right) = \sum_{i=1}^{N} \ln\left(\frac{\lambda_i}{1+\lambda_i}\right) = G \tag{7.12}$$

$$\left(\frac{\Delta V_b}{c}\right)_N = \sum_{i=1}^{N} \ln\left(\frac{1+\lambda_i}{\varepsilon+\lambda_i}\right) = F \tag{7.13}$$

Method of Lagrange Multipliers = method for finding max or min of a function.

Let

$$L = F + \alpha G \tag{7.14}$$

Where α is a constant (Lagrange Multiplier Constant)

We want to maximize F (deltaV) subject to constraint G (mass and payload ratios). L is maximized when F and G maximized.

$$\frac{\partial L}{\partial \lambda_i} = 0 \tag{7.15}$$

$$\frac{\partial L}{\partial \lambda_i} = \frac{\partial F}{\partial \lambda_i} + \alpha \frac{\partial G}{\partial \lambda_i}$$

$$= \frac{\partial}{\partial \lambda_i} \sum \ln\left(\frac{1+\lambda_i}{\varepsilon+\lambda_i}\right) + \alpha \frac{\partial}{\partial \lambda_i} \sum \ln\left(\frac{\lambda_i}{1+\lambda_i}\right) = 0 \tag{7.16}$$

$$= \sum_{i=1}^{N} \left[\frac{\partial}{\partial \lambda_i} \ln\left(\frac{1+\lambda_i}{\varepsilon+\lambda_i}\right) + \alpha \frac{\partial}{\partial \lambda_i} \ln\left(\frac{\lambda_i}{1+\lambda_i}\right)\right] = 0$$

So for this summation to be equal to 0, each term in this must be 0. Therefore

$$\frac{\partial}{\partial \lambda_i} \ln\left(\frac{1+\lambda_i}{\varepsilon+\lambda_i}\right) + \alpha \frac{\partial}{\partial \lambda_i} \ln\left(\frac{\lambda_i}{1+\lambda_i}\right) = 0$$

$$\frac{\partial}{\partial \lambda_i}\Big(\ln(1+\lambda_i) - \ln(\varepsilon+\lambda_i)\Big) + \alpha \frac{\partial}{\partial \lambda_i}\Big(\ln(\lambda_i) - \ln(1+\lambda_i)\Big) = 0$$

$$\left[\frac{\varepsilon-1}{(1+\lambda_i)(\varepsilon+\lambda_i)}\right] + \alpha\left[\frac{1}{\lambda_i\,(1+\lambda_i)}\right] = 0$$

So

$$\lambda_i = \frac{\alpha\,\varepsilon}{1-\alpha-\varepsilon} = \text{constant} \tag{7.17}$$

This is what the payload ratio should be to get the maximum ΔV for a multi stage N-stage rocket. Each stage should have the same payload ratio.

Best approach is $\lambda_i = \text{constant} \; = \lambda$, in other words, want each stage to have same payload ratio. Want Similar Stages., $\varepsilon, c/I_{SP}, \lambda_i$

2.5 Delta V for N-Stages

We found max ΔV is when $\lambda_i = \text{constant} = \lambda$

Then from (7.10)

$$\frac{(M_o)_N}{(M_o)_1} = \prod_{i=1}^{N} \left(\frac{\lambda_i}{1+\lambda_i}\right) = \left(\frac{\lambda}{1+\lambda}\right)^N \tag{7.18}$$

So,

$$\lambda = \frac{\left[\frac{(M_o)_N}{(M_o)_1}\right]^{\frac{1}{N}}}{1 - \left[\frac{(M_o)_N}{(M_o)_1}\right]^{\frac{1}{N}}} \tag{7.19}$$

Recall from Equation (7.9), we can plug in the optimum λ

$$(\Delta v_b)_N = c \sum_{i=1}^{N} \ln\left(\frac{1+\lambda_i}{\varepsilon+\lambda_i}\right) - g\,T_b$$

Such that

$$(\Delta v_b)_N = c\,N \ln\left(\frac{1+\lambda}{\varepsilon+\lambda}\right) - g\,T_b \tag{7.20}$$

With (7.19), this is:

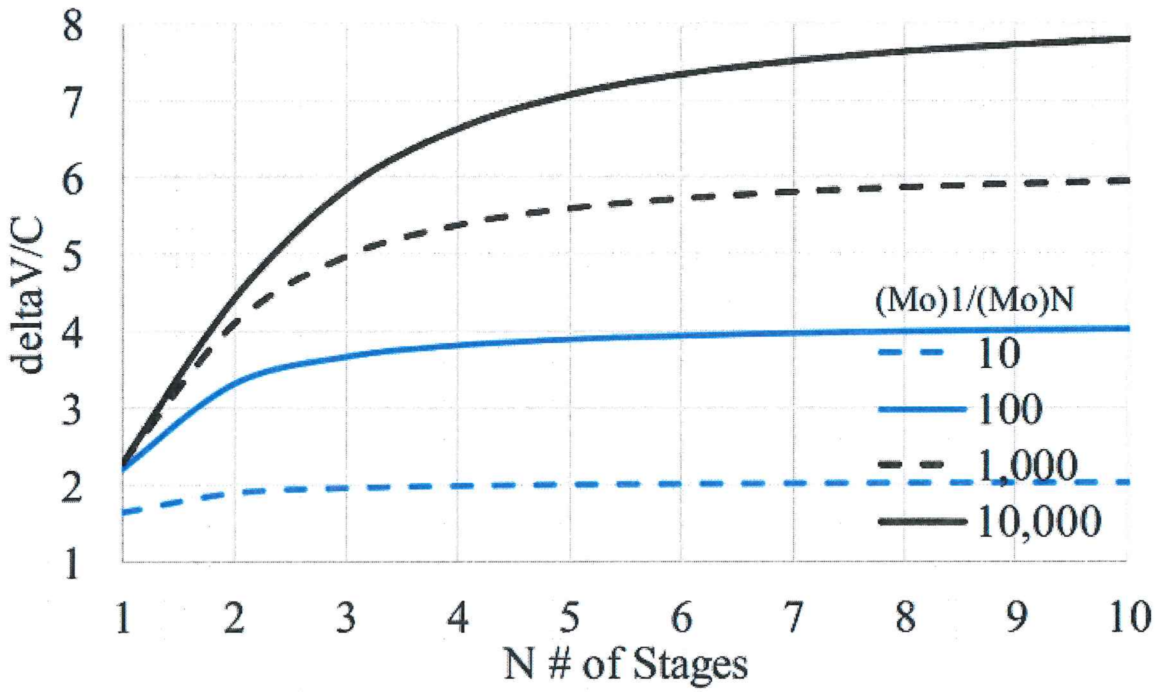
$$(\Delta v_b)_N = c\,N \ln \frac{1}{\varepsilon + (1-\varepsilon)\left(\frac{(M_o)_N}{(M_o)_1}\right)^{\frac{1}{N}}} - g\,T_b \tag{7.21}$$

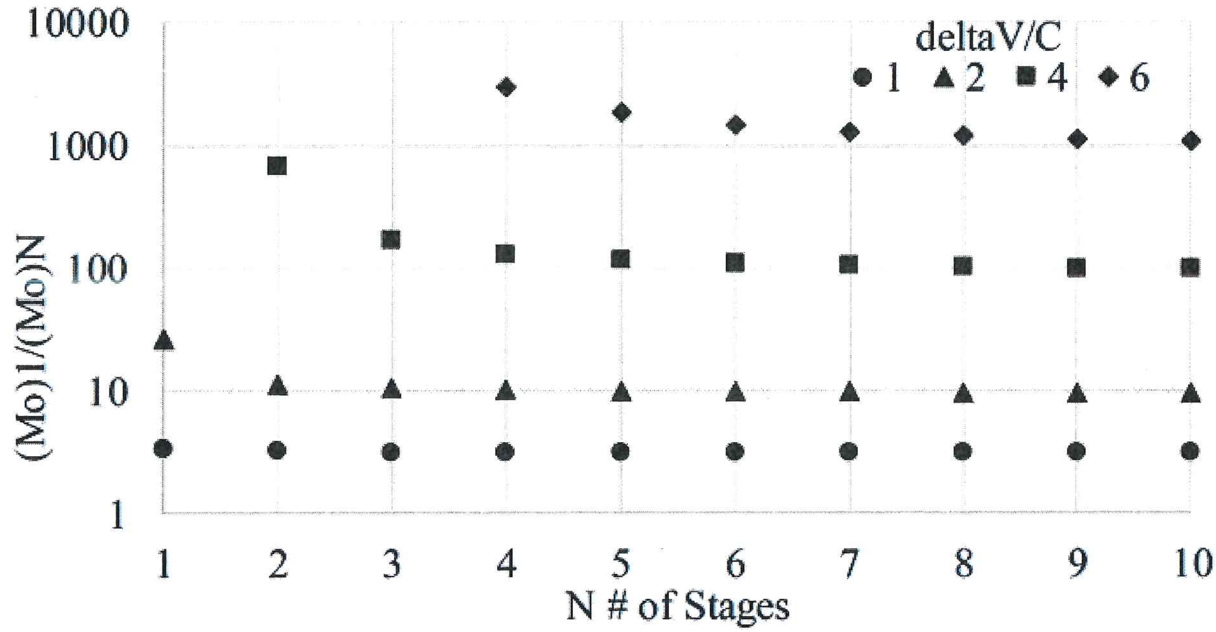
How sensitive is this to N? As $N \to \infty$:

$$\lim_{N\to\infty} (\Delta v_b)_N = c\,(1-\varepsilon) \ln\left(\frac{(M_o)_1}{(M_o)_N}\right) - g\,T_b \tag{7.22}$$

N^{th} stage (last stage) $(m_o)_N = \text{payload}_N + \text{structure}_N + \text{propellant}_N$

Here we see Eqn (7.21) without gravity ($\varepsilon = 0.10$):





So, depending on your mission adding stages decreases required mass ratio, but only to a point, it plateaus!

Example: Compare Single-Stage vs. Two-Stage Rocket

Consider a rocket with Isp = 310.7 sec. This results in c=3048 m/s. In addition, $m_L = 1000\,kg$, $m_o = 15000\,kg$, $m_{st} = 2000\,kg$. Compare the ΔV for...

For a single stage rocket:

$$\varepsilon = \frac{m_{st}}{m_o - m_L} = 0.143$$

$$\lambda = \frac{m_L}{m_o - m_L} = 0.0714$$

$$MR = \frac{1}{R} = \frac{1}{5}$$

$$V_b = c\,\ln\left(\frac{1+\lambda}{\varepsilon+\lambda}\right) = 4905\,m/s$$

↑ From (7.9) with no gravity

For a multi-stage rocket:

The stages are identicle so $\varepsilon =$ const. and $\lambda =$ const.

$$\lambda = \lambda_1 = \lambda_2 = \frac{m_{o,2}}{m_{0,1} - m_{0,2}} = \frac{m_L}{m_{0,2} - m_L}$$

From this we find that $M_{0,2} = 3873\,kg$ and $\lambda = 0.348$

$$\varepsilon_1 = \varepsilon_2 = \varepsilon = \frac{m_{st,1}}{m_{0,1} - m_{0,2}} = \frac{m_{st,2}}{m_{0,2} - m_L}$$

$$M_{st,1} + M_{st,2} = 2000\,kg$$

From this we find that $M_{st,1} = 1589kg$ and $M_{st,2} = 411kg$ and $\varepsilon = 0.143$.

$c_1 = c_2 = c$ Equation 7.20 yields w/ no gravity

$$V_b = 2\,c\,\ln\left(\frac{1+\lambda}{\varepsilon+\lambda}\right) = 6157\,m/s$$

Section 2: Rockets Basics

AE435

Spring 2018

3 Thrust Chambers

Contents

3.1 Introduction

Chemical rockets consist of a propellant supply and feed system, combustion chamber and nozzle.

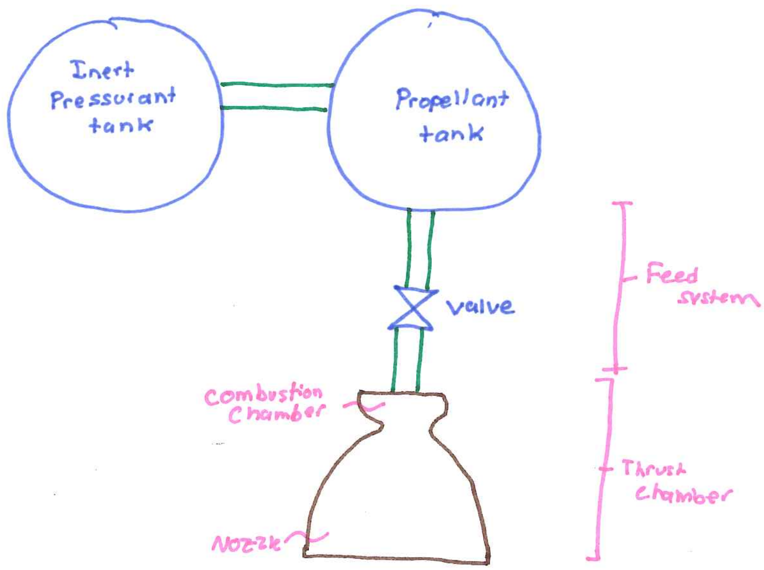


Figure: Propellant Feed System

To simplify our analysis, assume:

1. Working fluid is a perfect gas, constant composition (not reacting)
2. Chemical reaction equivalent to a constant pressure heating process
3. Expansion is steady, 1-D, isentropic flow

3.2 Performance Characteristics of Ideal Rocket

To simplify our analysis, assume:

1. Working fluid is a perfect gas, constant composition (not reacting)
2. Chemical reaction equivalent to a constant pressure heating process
3. Expansion is steady, 1-D, isentropic flow

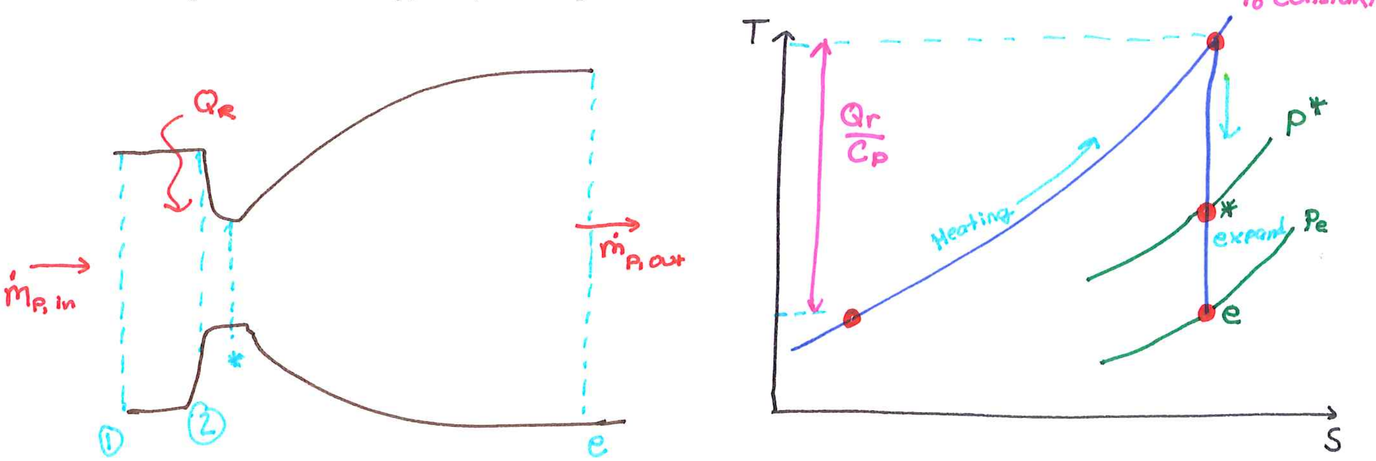


Figure: Schematic of a Rocket Engine

- State (1) - Propellant input at temperature T_1
- State (2) - Combustion has occurred, propellant constant pressure heated to temperature T_2, heating value Q_r.
- (*) to (e) - propellant expands through the nozzle throat (*) to the exit (e).

Energy Equation:

Heating Rate: $$\dot{Q} = \dot{m}\,(h_{o,2} - h_{o,1}) = \dot{m}\,C_P\,(T_{o,2} - T_{o,1}) \tag{8.1}$$

$$T_{o,2} = T_{o,1} + \frac{Q_r}{C_P} \tag{8.2}$$

Now we have adiabatic nozzle expansion. What this means is that the total enthalpy is conserved. $h_{o,2} = h_{o,e}$

In this chemical rocket, the heat is coming from the chemical reaction. Causing the total temperature to increase. The nozzle is adiatbative so we assume no heat addition or loss therefore the enthalpies must be the same.

$$h_{o,2} = h_e + \frac{u_e^2}{2} \quad \longrightarrow \quad \frac{u_e^2}{2} = C_P\left(T_{o2} - T_2\right)$$

$$u_e = \sqrt{2\,C_P\,T_{o,2}\left(1 - \frac{T_e}{T_{o,2}}\right)}$$

Earlier when we had this equation, we were making the assumption that the exit temperature is much much more than the input temperature.

Assuming Isentropic relations

$$u_e = \sqrt{2\,C_P\,T_{o,2}\left(1 - \left(\frac{P_e}{P_{o,2}}\right)^{\frac{\gamma-1}{\gamma}}\right)} \tag{8.3}$$

Lets modify this by using $C_P = \frac{\gamma\,\mathscr{R}}{(\gamma-1)\,MW}$

Now we can write

$$u_e = \sqrt{\frac{2\,\gamma\,\mathscr{R}}{(\gamma-1)\,MW}\,T_{o,2}\left[1 - \left(\frac{P_e}{P_{o,2}}\right)^{\frac{\gamma-1}{\gamma}}\right]} \tag{8.4}$$

Advantageous to use propellants with low molecular weight to keep u_e high. If we want high u_e we want our pressure ratio to be as low as possible, low molecular weight and large combustion temperature.

With (8.2), then

$$u_e = \sqrt{\frac{2\,\gamma\,\mathscr{R}}{(\gamma-1)\,MW}\left(T_{o,1} + \frac{Q_r}{C_P}\right)\left[1 - \left(\frac{P_e}{P_{o,2}}\right)^{\frac{\gamma-1}{\gamma}}\right]} \tag{8.5}$$

We also want a really large heating values, Q_r to be large and P_{o2} to be large too.

Often $\frac{Q_r}{C_P} \gg T_{o,1}$

$$u_e = \sqrt{\frac{2\gamma\mathscr{R}}{(\gamma-1)\,MW}\left(\frac{Q_r}{C_P}\right)\left[1-\left(\frac{P_e}{P_{o,2}}\right)^{\frac{\gamma-1}{\gamma}}\right]} \tag{8.6}$$

The mass flow rate through the chocked nozzle is given as:

$$\dot{m} = \frac{A^* P_{o,2}}{\sqrt{R T_{0,2}}}\sqrt{\gamma\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}} \tag{8.7}$$

Using the thrust equation:

$$F = \dot{m}\,u_e + (P_e - P_a)\,A_e \tag{8.8}$$

With (8.7) and 8.3

Thrust Coefficient

$$C_F = \frac{F}{A^*\,P_o} = \sqrt{\frac{2\,\gamma^2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}\left[1-\left(\frac{P_e}{P_{o,2}}\right)^{\frac{\gamma-1}{\gamma}}\right]} + \left(\frac{P_e}{P_o} - \frac{P_a}{P_o}\right)\frac{A_e}{A^*} \tag{8.9}$$

Where

$P_0 = P_{02}$ = combustion chamber pressure

P_a = ambient pressure

To further analyze and compare rockets, using this ideal analysis, we use the characteristic velocity (metric for combustion chamber) and thrust coefficient (metric for nozzle).

3.2.1 Characteristic Velocity

$$c^* = \frac{P_o\,A^*}{\dot{m}} \tag{8.10}$$

Using (8.7) we get

$$c^* = \sqrt{\frac{1}{\gamma}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}\frac{\mathscr{R}\,T_o}{MW}} \tag{8.11}$$

$T_0 = T_{02}$ = combustion chamber temperature, total temperature

Thus c^* is only a property of combustion chamber properties. c^* depends on fuel and oxidizer, see table.

TABLE 11.1 Vacuum expansion p_{01} = 1000 psia (6.89 MPa); $\frac{A_e}{A^*} = 40$

Oxidizer	LO_2	LO_2	LO_2
Fuel	LH_2	RP1 ($CH_{1.97}$)	CH_4
Oxidizer–fuel mass ratio	4.83	2.77	3.45
T_{01}/K	3250	3700	3560
Average bulk density of propellant, kg/m^3	320	1030	830
I_{sp}	455	358	369
u_e m/s	4464	3512	3620
c^* m/s	2386	1838	1783

Source: Data from Rocketdyne Chemical and Material Technology, Rocketdyne Division, Rockwell International.

Combustion temperature varies by $\approx 14\%$, but MW of products (avg. bulk density) varies by factor of > 3! . Of course smaller MW gives rise to higher u_e and c^*. c^* vary for each propellant combination, here by $\approx 34\%$.

Metric used for a combustion chamber is often the c^*-efficiency:

c* Efficiency

$$\eta_{c^*} = \frac{c^*)_{actual}}{c^*)_{theoretical}}$$

3.3 Thrust Coefficient

$$C_F = \frac{F}{P_o A^*} \tag{8.12}$$

The amplification of the thrust. How the nozzle is amplifying the thrust of the flow above what you'd get from just pressure acting on the throat.

This represents the amplification of thrust due to gas expansion in nozzle compared to pressure on throat area alone. Thrust coefficient is the amplification coefficient provided by the nozzle.

C_F is therefore a function of nozzle geometry, measure of how well nozzle geometry suited to actual pressure ratio.

C_F is maximum (optimum) when $p_e = p_a = p_\infty$, called a "matched" nozzle.

A measure of how well the nozzle is matched to the ambient conditions. Whenever the design rpessure ratio = the actual pressure ratio is when it is "matched"

Combining (8.12) and (8.10) together $\rightarrow$

$$F = \dot{m}\, c^*\, C_F \tag{8.13}$$

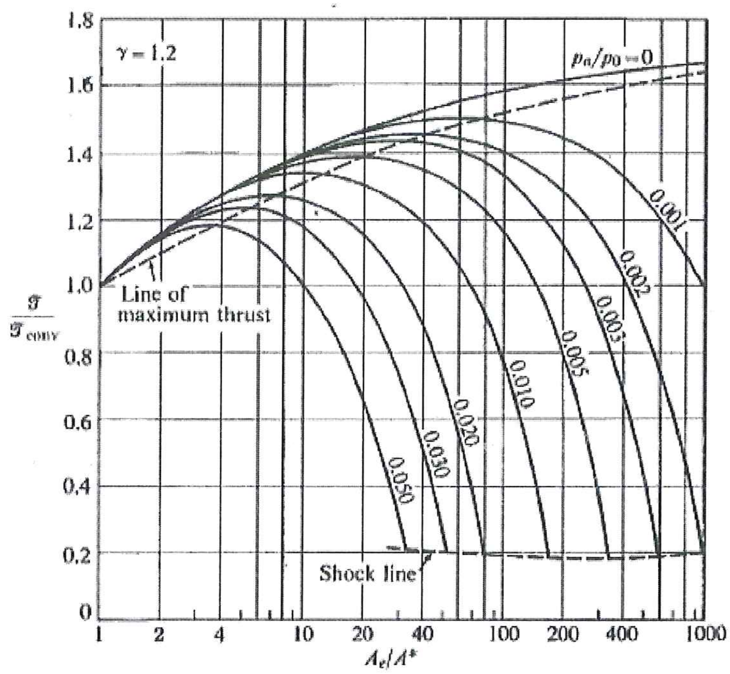


FIGURE 11.3 Performance characteristics of a one-dimensional isentropic rocket nozzle; $\gamma = 1.2$. (After Malina [1].)

Thrust coefficient as a function of nozzle area ratio for different ambient pressures.

- "line of maximum thrust" goes through the peak of each pressure curve. And is the corresponding area ratio that provides pressure ratio identical to that of the background pressure.
- Larger area ratio is required to expand to lower background pressure, p_a.
- Having too large an area ratio can actually reduce thrust
- "shock line" indicates the pressure ratios and area ratios for which a shock enters the nozzle.

This figure is showing thrust coefficient (y-axis) as a function of the nozzle area ratio. Each curve represents a different ambient pressure ratio. We should recognize that there is a maximum in these curves. For a given ambient pressure, there is an optimum area ratio that optimizes the are ratio. Another thing to recognize is that at lower ambient pressures, you want a larger area ratio. Also if we have a poor nozzle, we will see that the nozzle is actually subtracting from the thrust. This is because we have over expansion , the area ratio is to big. If our C_F is less than one, it means we are over expanded and we have shock waves. . The shock lines means we have a stationary shock at the exit of the engine.

Typically, we design thruster engines such that it is a little bit over expanded at sea-level but changes as we move to higher altitudes.

3.4 Nozzles

Figures of Merit:

1. Maximize C_F, thrust
2. Minimize stagnation pressure loss, no shocks, minimize boundary layer, smooth surface
3. Short and light as possible
4. Ease of manufacturing
5. Cooling, self-adjusting
6. Liquid vs. solid propellant. Solid particulates are common in a solid rocket motor plume. Two phase flow of both gaseous products and microscopic solid particles which will erode or ablate nozzle surfaces.

3.4.1 Conical Nozzles

Larger rockets, have parabolic nozzle. Smaller have conical nozzle.

Basis from which more advanced, more complex parabolic "bell" nozzles are developed.

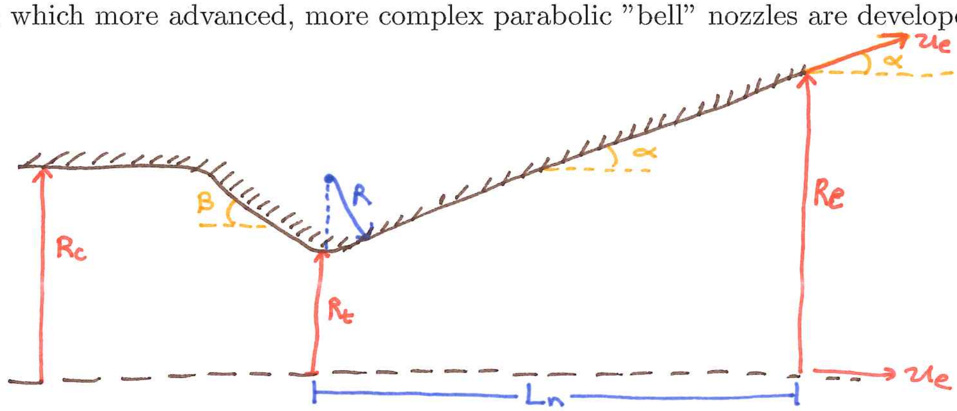


Figure: General Geometry of a Conical Nozzle

Where

$$R_c = \text{radius of combustion chamber}$$
$$R_t = \text{throat radius}$$
$$R_e = \text{radius of nozzle exit}$$
$$R = \text{"throat circle" radius}$$
$$L_n = \text{nozzle length from throat to exit}$$
$$\alpha = \text{angle of diverging section}$$
$$\beta = \text{angle of converging section}$$

α -Angle of diverging section , Typically $\alpha < 15°$. Generally not $> 15°$ because then we'd get flow separation and loss!

β - Angle of converging section, not critical because flow is not moving as fast. Typically between $25° < \beta < 45°$

Conical nozzles (in fact all nozzles) have non-axial flow velocities. If youd imagine what the flow looks like, along center line, it has a nice axial velocity. But as you move out to the wall, u_e is not entirely axial, in fact, it will have an angle α. As a result, not all the momentum is not in the axial direction and we must take this into account.

How to account for this effect?

C.V. analysis shows that the thrust of an ideal conical rocket of half-angle α is reduced by a factor, λ (not the payload ratio!): Sometimes referred to as the cosine loss.

Thrust Correction Factor

$$\lambda = \frac{1 + \cos(\alpha)}{2} \tag{8.14}$$

Actual thrust is then

$$F_{\text{actual}} = \lambda F_{\text{ideal}} \tag{8.15}$$

Data table of Eqn. (8.14)

TABLE 3-3. Nozzle Angle Correction Factor for Conical Nozzles

Nozzle Cone Divergence Half Angle, α (deg)	Correction Factor, λ
0	1.0000
2	0.9997
4	0.9988
6	0.9972
8	0.9951
10	0.9924
12	0.9890
14	0.9851
15	0.9830
16	0.9806
18	0.9755
20	0.9698
22	0.9636
24	0.9567

From the geometry of the conical nozzle figure above:

$$L_n = \frac{R_t\left(\sqrt{\varepsilon}-1\right) + R\left(\sec(\alpha)-1\right)}{\tan\alpha} \tag{8.16}$$

We can make the approximation...

$$L_n \approx \frac{R_t\left(\sqrt{\varepsilon}-1\right)}{\tan\alpha} \tag{8.17}$$

because our circle radius is much smaller than our throat radius.

ε - Nozzle area ratio (A_e/A_t)

Recognize that eqn. (8.16) and (8.17) are the same for both upstream of the throat, and downstream of the throat. Specifically,

- $L_n(\alpha)$- Length throat-to-exit. (Downstream)
- $L_n(\beta)$ - Length combustor to throat (replace α with β and use ε_c) (Upstream)
- ε_c - Contraction ratio ratio (A_c/A_t)

Note:

$$R_c = \sqrt{\varepsilon_c}\,R_t$$

$$R_e = \sqrt{\varepsilon}\,R_t$$

The length of a conical nozzle with $\alpha = 15°$ is used as reference length for a given ε.

That is: $L_n(\alpha = 15°)$ = reference length for a given ε.

Fractional Length

$$L_f = \frac{\text{Actual Nozzle Length}}{\text{Reference Length}} \times 100\% \tag{8.18}$$

3.4.2 Bell Nozzle

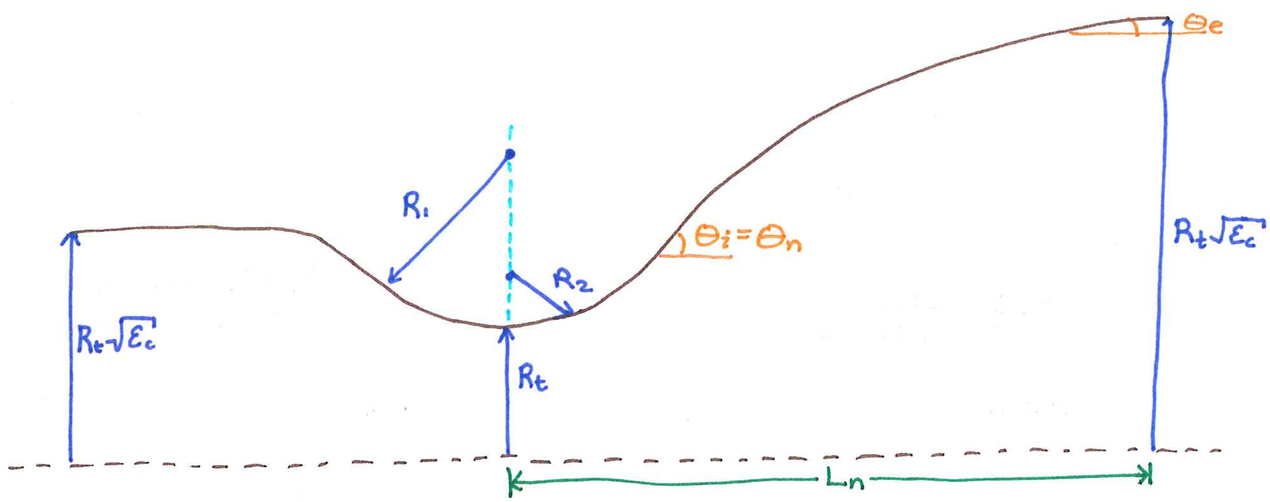


Figure: General Geometry of a Bell Nozzle

Where

$$R_1 = \text{upstream arc radius}$$
$$R_2 = \text{downstream arc radius}$$
$$\theta_i = \text{initial angle of the diverging downstream nozzle}$$
$$\theta_e = \text{final angle of the diverging downstream nozzle}$$

Bell Nozzles have

- Higher rate of expansion to lower rate of expansion.
- Generally shorter nozzle, saves weight and less boundary layer effects.

Why does a bell nozzle start with a higher rate of expansion to lower rate of expansion. As you move to the right, the area is increasing rapidly. But as we get to the end, the area does not change as much. We get a higher rate of expansion right outside the throat and the rate of expansion decreases as we get closer to the exit.

Bell nozzles are generally shorter which saves weight and also less boundary layer effects because there is less wall length. Therefore less irreversibility and less viscous effects.

Concerns:

- Turn too fast, the flow may freeze: i.e. θ_i is to large, this results in the flow freezing. This means that the composition is not changing anymore. The flow turns so fast that the composition does not have time to change.
- Turn too fast, the solid particulates just run into the wall (from solid booster). Solid particles hit the wall and can erode and create hotspots in the nozzle.

Bell nozzles also have cosine loss.

Relationship between L_f and α of a conical nozzle, and θ_e and θ_i for a bell nozzle:

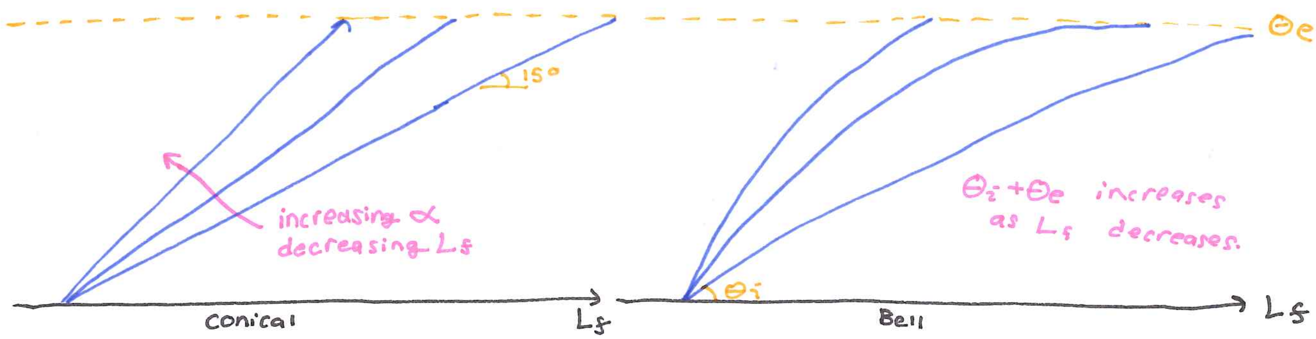


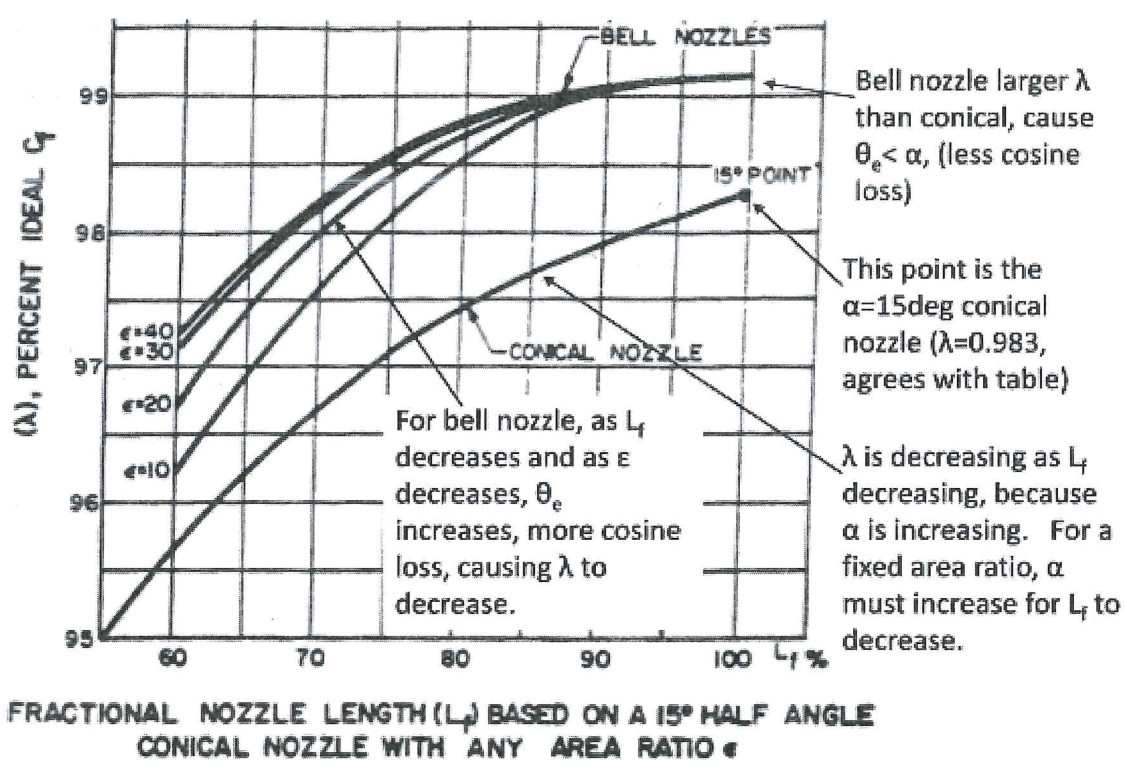
Figure: Conical V. Bell Nozzle

Conical Nozzle: For fixed area ratio, as L_f decreases, we see geometry requires α to increase

Bell Nozzle: For fixed area ratio, as L_f decreases, we see geometry requires θ_e and θ_i to increase.

As the angles increase, gives rise to more cosine loss

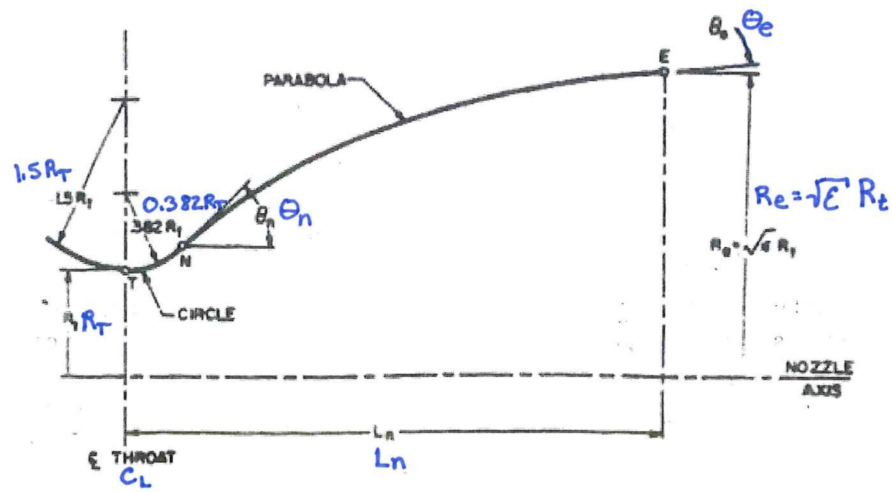
Handout 2H illustrates this (below)



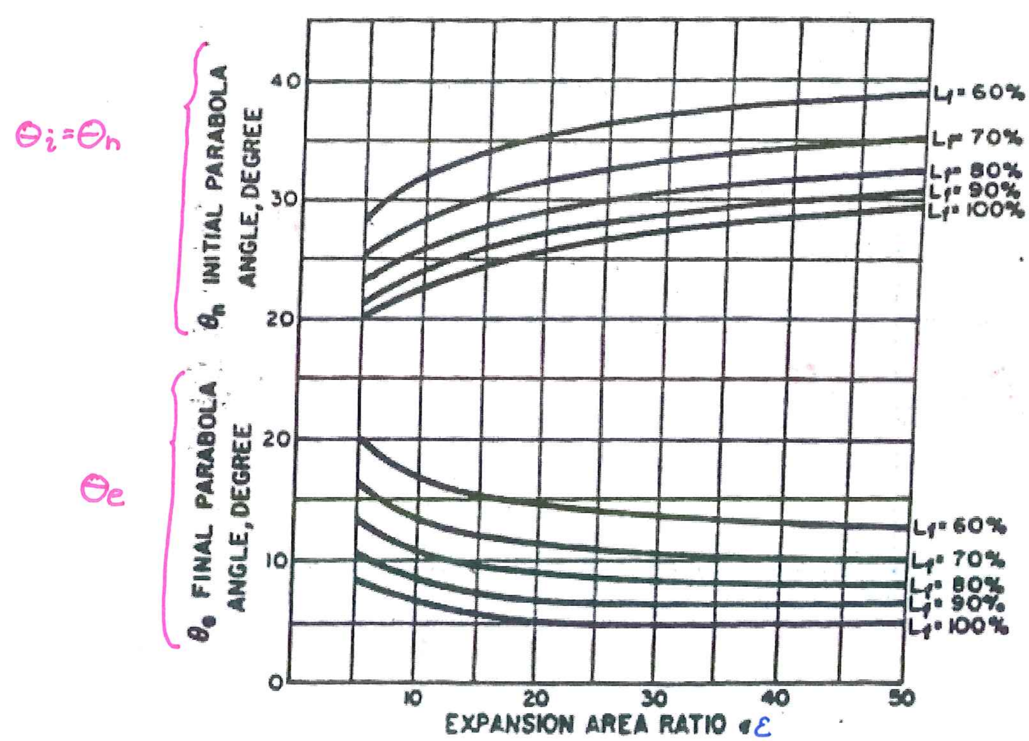
3.4.3 Rao's Method for Bell Nozzle Design

Rao's method is semi-empirical and based on numerical simulations using the method of characteristics to maximize thrust. Assumes a parabolic diverging section.

Rao found the following optimum geometry:



Note the upstream throat circle radius of $1.5R_t$, and downstream throat circle radius of $0.382R_t$. The optimum initial ($\theta_i = \theta_n$) and final (θ_e) angles are given in the figure below.



Example: Consider a rocket with: 750, 000 lbf, Isp =270sec, $R_t = 12.45$", $L_f = 80\%$, $\varepsilon = 14$, $\beta = 20°$, $\varepsilon_c = 1.6$ Contraction Ratio,

Method of Rao, $R_{\text{upstream}} = 1.5\, R_t$, $R_{\text{downstream}} = 0.382\, R_t$

What should the length of inlet (converging) and diverging sections be? What should the shape of the nozzle be?

Inlet Length

$$L_i = \frac{12.45\,(\sqrt{1.6} - 1) + 1.5\,12.45\,(\sec(20°) - 1)}{\tan 20°} = 12.4"$$

Diverging Length:

$$L_d = 0.8L(\alpha = 15°, \varepsilon = 14)$$

14   $0.382 \cdot R_t$

$$= 0.8\,\frac{12.45\,(\sqrt{\cancel{1.6}} - 1) + \cancel{1.5\,12.45}\,(\sec(20°) - 1)}{\tan 20°} = 102"$$

What about our angles?

$$L_f = 80\%, \qquad \varepsilon = 14 \qquad \longrightarrow \qquad \theta_i = \theta_n \cong 27° \qquad \theta_e = 10°$$

We now have the lengths, but now how about the shape of the diverging section??

Fit a parabola between the start and end of the diverging section.

1. You can find the start and end points of the diverging section:

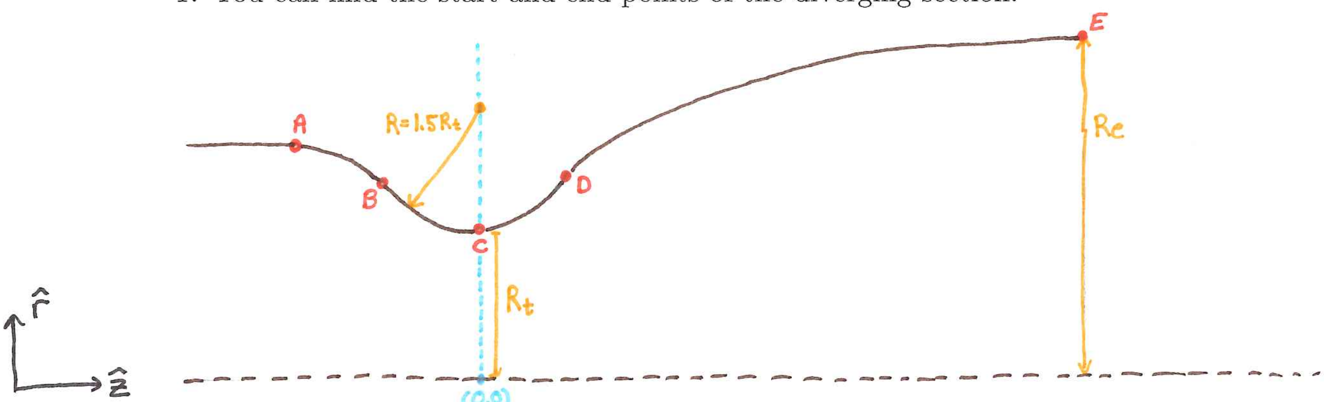


Figure: Optimizing Bell Nozzle

For HW, set your origin at the throat, Point C.

2. From this figure we see that we can calculate the points A,B,C,D and E by:
   - 2 A: From L_i and $R_c = \sqrt{\varepsilon_c}\, R_t$
   - 4 B: Circle at 2.5 R_t with radius $R = 1.5\, R_t$, match slope at 20°
   - 1 C: Throat Radius and Origin
   - 3 D: Circle at 2.5 R_t with radius $R = 0.382\, R_t$, match slope at $\theta_i = \theta_n$
   - 5 E: From L_d and $R_e = \sqrt{\varepsilon}\, R_t$

3. Fit D to E with a parabola. In particular you can do:

$$AA\, r^2 + BB\, z + CC\, r + DD = 0$$

4. We need 4 pieces of info to get AA,BB,CC,DD.

   We know the location of point D and E. We also know the slopes at D and E

5. This Rao method provides the maximum thrust for a nozzle of given length (to within a few %), assuming a parabolic profile. This is an analytical technique (doesn't require much time) based on semi-empirical results from numerical sims. More sophisticated fluid dynamic sims (e.g., Fluent) can be used to further optimize the nozzle geometry (get more thrust), but this technique of Rao gets you 90% of the answer with only 10% of the effort.

3.4.4 Nozzle Backpressure Effects

- A pressure difference must exist to force gas through a nozzle.
- $P_e < P_i$, or if Pi is the total pressure of the upstream combustion chamber, $P_i = P_o$, and $P_e/P_o < 1$

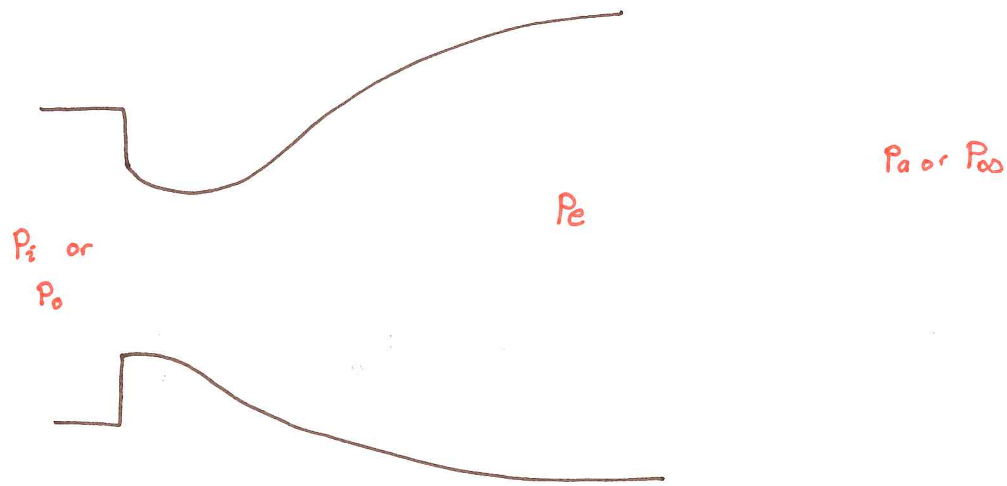


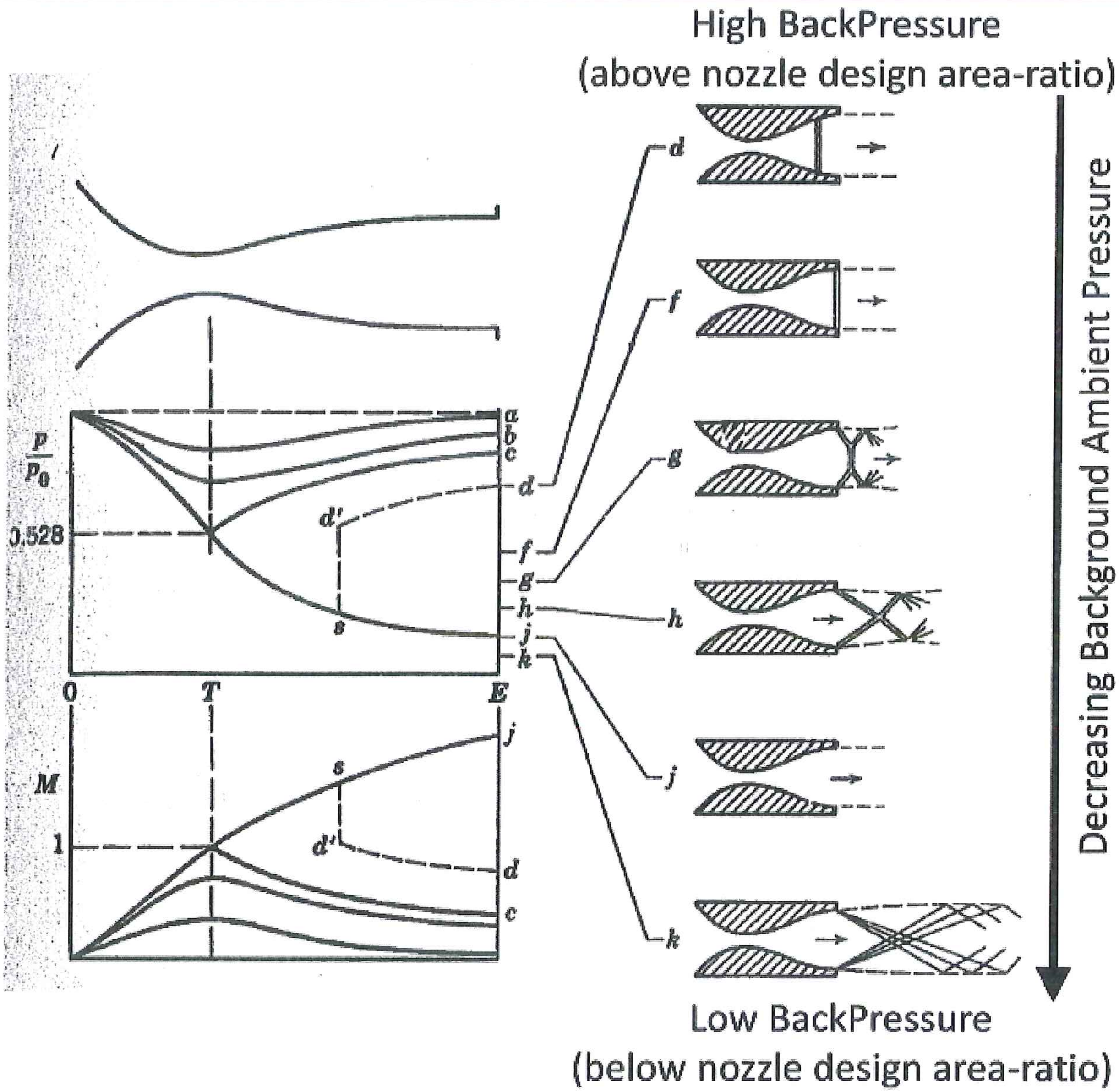
Figure: Conical V. Bell Nozzle

As soon as you have selected the area ratio, you have selected the nozzle pressure ratio. By selecting the area ratio, you select the exit mach number which has a resulting pressure ratio.

The design pressure ratio may not give you the same exit pressure that will not match the ambient background. We must understand how the flow behind the nozzle changes as a result

When you pick a nozzle area ratio,ε , you have selected the nozzle pressure ratio P_e/P_o (see Eqn. 4.110, area ratio Mach # relation, and then Eqn. 4.14, isentropic pressure ratio Mach # relation).

What happens if the exit pressure the nozzle provides P_e, based on its fixed P_e/P_o, does not match the ambient pressure Pa? Note this is always the case in space where $P_a = 0$. Liepmann and Roshko (Fig 5-3)



- A) Ambient pressure just barely below the upstream total pressure. Gas expands to max velocity at throat (but stays low subsonic). Pressure decreases to min at throat, then increases to match the ambient pressure at the exit.
- B) Ambient pressure a little lower, so gas expands to higher velocity maximum at throat (but still subsonic). Pressure decreases to min at throat, then increases to match the ambient pressure at the exit.
- C) Ambient pressure still lower, and now it's low enough to just achieve M = 1 at the throat, but then the flow goes back to subsonic, pressure increases to match the ambient pressure at the exit.
- D) Ambient pressure still lower, achieves M =1 at throat, and get supersonic flow downstream of throat. BUT, the area ratio of nozzle is too large for this ambient pressure. If allowed to

continue the nozzle will expand the gas to way too low a pressure (much below the ambient). So a gasdynamic structure (shockwave) forms to increase the pressure (change to subsonic M#) such that the expansion matches the ambient pressure. Shockwave in the nozzle? .very BAD. In this case, the shock forms farthest upstream in the diverging section. (Overexpanded)

- F) Ambient pressure still lower, but nozzle still trying to expand the flow too much (area ratio too big), shock wave must still form to increase pressure and match the exit and ambient pressures. Shock has moved downstream and now stands at the the nozzle exit. (Overexpanded)
- G) Ambient pressure low enough that now the shock has moved outside the nozzle, and oblique shocks are attached to the exit of the nozzle. These gas dynamic structures form in order to alter the pressure of the exiting gas such that it eventually matches the ambient. These are "shock diamonds" or "mach diamonds" (Overexpanded)
- H) Lower ambient pressure, shock diamonds have expanded in size. (Overexpanded)
- J) Ambient pressure still lower, low enough now that the design exit pressure of the nozzle matches the ambient pressure exactly. No need for gas dynamic structures (shocks) to match the pressure, it's already matched. (MATCHED)
- K) Ambient pressure still lower, in fact, lower than the designed exit pressure of the nozzle. Again we need a gas dynamic structure to change the exit pressure to match the ambient pressure. Now that structure is an expansion wave (the pressure needs to be decreased). (Underexpanded)

Rocket nozzles are designed to generally start at around state H at sea-level (never D or F, very violent to have shock inside nozzle, plus, exhaust velocity is SUBSONIC!), and then as they climb, the backpressure decreases, so they progress to J and eventually K at high-altitude and in space.

Altitude compensation techniques (two-step nozzles), but these are rare (ground test experience only or limited flight tests):

- Add Extendible nozzle segment to increase the area ratio as you increase in altitude.
- Add an Insert, then eject it as you climb in altitude to increase area ratio.
- Add a ring-shaped bump to your nozzle, at sea level, flow expands up to and separates at the bump, as altitude increases, gas eventually expands into the entire volume and separates at the exit

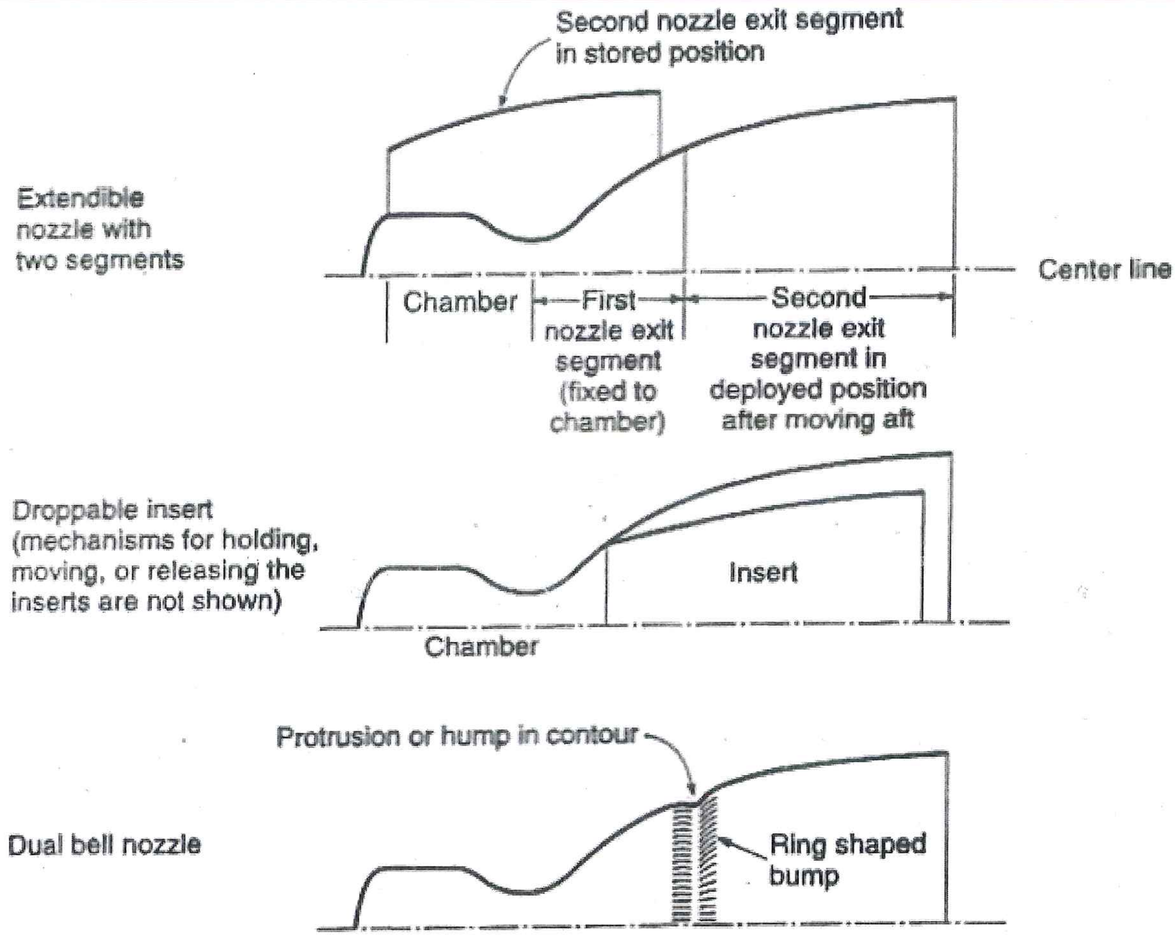


FIGURE 3–15. Simplified diagrams of three altitude-compensating two-step nozzle concepts.

3.4.5 Plug and Expansion-Deflection

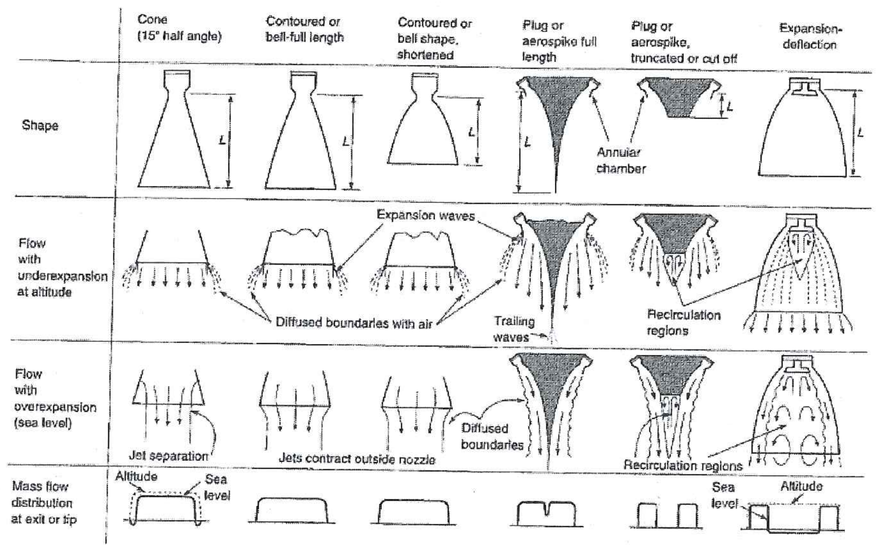
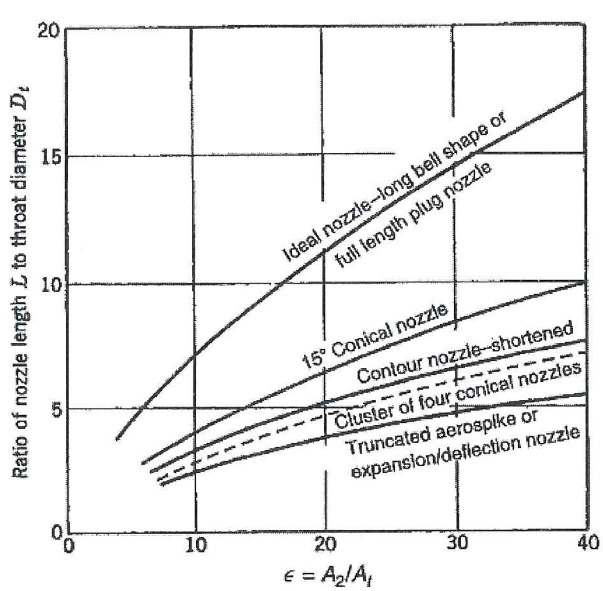


FIGURE 3–12. Simplified diagrams of several different nozzle configurations and their flow effects.

- Comparison conical, bell, and altitude compensating nozzles (plug/aerospike and expansion deflection).
- Altitude compensating nozzles have their expansion dictated and controlled by the ambient atmosphere.
- No altitude compensating nozzles have flown in a production vehicle (yet). Firefly Aerospace was planning (2014), but appears to have scrapped now for bell nozzle. Will Haas 2CA be the first with linear aerospike engine?



Section 2: Rockets Basics

AE435

Spring 2018

3 Rocket Propellants

We will talk about liquid propellants first,

Then frozen vs. equilibrium flow

Then solid propellant analysis.

Contents

3.1 Liquid Propellants

The **BOLD BLACK** propellants are the most commonly used today.

1. Monopropellant
   - Single substance capable of exothermic breakdown when ignited
   - Most common is hydrazine, considerable application in small rockets for attitude control

   (a) Hydrazine (N2H4)
   - Storable for long periods
   - Good for short-duration, pulsed operation
   - Catalyst is iridium
   - Theoretically Isp 240sec, pulsed operation lowers this due to dynamic effects and thermal transient effects
   - Toxic, colorless liquid, freezes at 273K
   - Volatile, very very toxic.

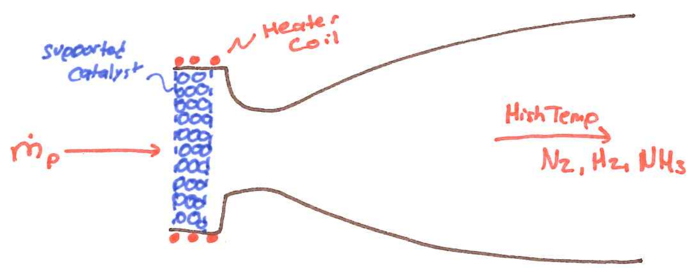


Figure: Hydrazine Thrust Chamber

   (b) Hydrogen peroxide (H2O2)
   - Used in some engines before 1955 (X-1 and X-15 aircraft)
   - Less use today

   (c) Ethylene Oxide and Nitromethane
   - Tried experimentally, no longer today

2. Bi-propellant
   - Fuel and oxidizer are separate

3.1.1 OXIDIZERS

1. **LOX (Liquid O2)**
    - Widely used
    - Good performance with alcohols (V2 and Redstone), jet fuels (kerosene, Atlas, Titan, Saturn), gasoline, and hydrogen (SSME, Delta), lots of fuels available
    - Hard to store, cryogenic, boils at 90 K
    - Explosion and fire possible under pressure
    - Necessary to insulate all lines, tanks, valves to reduce evaporation losses
2. Hydrogen Peroxide (H2O2)
    - Highly concentrated form 70-99% (rest H2O), commercial grade (Walgreens) maybe 30% at most
    - Hypergolic (spontaneously ignites) with hydrazine and burns well with kerosene
    - Decomposes over time, hard to store, releases heat as decomposes, explosion hazard at 448 K
    - Not much use today
3. Nitric Acid (HNO3)
    - Not much use today
    - Most common RFNA, red fuming nitric acid
    - Red-brown poisonous fumes
    - Also WFNA (white fuming)
    - Highly corrosive (only SS and gold for storage)
    - Hypergolic with hydrazine
4. **Nitrogen Tetroxide (N2O4, NTO)**
    - Most common storable oxidizer in US use
    - Can be stored indefinitely in sealed containers
    - Hypergolic with many fuels and spontaneous ignition with paper, wood, leather
    - Reddish-brown fumes toxic
    - High vapor pressure means heavy tanks
    - Used in Titan missile, and SS OMS and RCS engines

3.1.2 FUELS

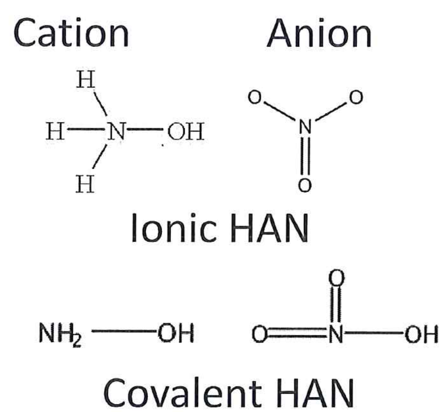
1. **Hydrocarbons**
    - Most can be used, gasoline, kerosene, diesel oil, turbojet fuel
    - RP-1 (rocket propellant 1) particularly good, kerosene-like mixture Atlas, Delta, Titan, Saturn, SpaceX Merlin, Antares
    - Methane (CH4), cryogenic, candidate for future boosters, RCS with LOX. Only experimental demonstration so far. Will SpaceX Raptor for BFR be first?
2. **Liquid Hydrogen (H2)**
    - High performance
    - Cryogenic, 40 K boiling point, well insulated tanks, valves, and also vacuum jacket for insulation
    - Mixture with air highly flammable and explosive so leakage vent line often ignited
    - LH2 and LOX on Centaur, SSME, Russia Europe, China upper stages, Delta IV
3. **Hydrazine (N2H4), Monomethylhydrazine (MMH), and Unsymmetricaldimethylhydrazine (UDMH)**
    - Toxic, colorless liquid, freezes at 273 K
    - Hypergolic with NTO and nitric acid
    - Explosive mixture with air
    - Stable liquid, excellent storability
    - Hydrazine with UDMH is more stable, lower freezing, higher boiling point, slightly lower Isp
    - Hydrazine UDMH 50% Titan missile and launch vehicle, lunar landing and takeoff engines (Apollo)
    - MMH with NTO in small attitude control systems
    - Better shock resistance, heat transfer and temperature range than hydrazine
    - High volatility, don't breath the vapors, respirator, significant PPE, limited exposure time

3.1.3 NEW PROPELLANTS

How do we replace, nasty hydrazine?

1. **Hydroxyl Ammonium Nitrate (HAN) (H4N2O4)**
    - Hygroscopic (likes to absorb moisture) solid when pure
    - Clear colorless odorless liquid in aqueous solution
    - Solid = solidpropellant, liquid = monopropellant

- Corrosive, toxic, denser than hydrazine liquid
- Not carcinogenic, low volatility, lower inhalation hazard
- Dangerous as solid, and in high concentration in solutions
  - Two forms: ionic and covalent
  - The proton transfer reaction is relatively low energy of 15 kcal/mol (0.65 eV/-molecule), suggesting that it may be possible for both ionic and covalent forms to exist together in solution
  - Nitric acid build-up, very reactive, dangerous, explosion hazard



- High temperature combustion >2000K, melt/sinter catalyst
- Aqueous solution
  - stabilizer (prevents, reduces nitric acid buildup)
  - Lower combustion temperature (1700K), no sinter catalyst

2. **New "Green" Propellants**

   (a) AF-M315E - HAN-based
   - Will it fly? NASA Green Propellant Infusion Mission (GPIM) - supposed to be 2016? then 2017? now 2018?
   - Ball Aerospace building spacecraft, Aerojet supplying rocket engines.

   (b) LMP-103 - Ammonium dinitrimide (ADN, H4N4O4, based)
   - Flown on Swedish PRISMA mission, formation flying with HPGP Sept. 2011
   - 6% higher Isp than hydrazine monoprop
   - 24% higher density (higher density-specific impulse, very good)
   - Low toxicity (ability to damage an organism)
   - Environmentally benign
   - Swedish space corporation - ECAPS company creates their high performance green propulsion (HPGP)

3.2 Frozen and Equilibrium Composition

- Isp depends significantly on composition and temperature of the combustion products
- At typical combustion pressure and temperature in rockets, the time for mixing of fuel and oxidizer and for chemical reactions is small compared to the residence time.

$$t_{\text{mix}} < t_{\text{combust}} < t_{\text{residence}} \tag{9.1}$$

- So an equilibrium assumption in the combustion chamber is justified (this was our prior combustion analysis with adiabatic flame temp, etc.)
- The temp of combustion products is high enough for some dissociation to be present (we've seen this in our equilibrium composition calcs)
- Would like for those dissociation products to recombine, release their energy back into the propellant stream.
- As product mixture expands out the nozzle, there may be insufficient time for dissociation products to recombine.
- Sometimes the rate of recombination is so slow that the gas composition effectively freezes and composition stays the same throughout nozzle.

So there are two extremes (actual rocket engine operates somewhere between these extremes):

- Equilibrium flow - composition of product mixture gas changes through the nozzle expansion, as gas cools, species recombine, release energy into flow, composition is always at equilibrium.
- Frozen flow - composition of product mixture gas does not change, stays constant (frozen) throughout the nozzle expansion

3.2.1 Dissociation Loss

Dissociation reduces the chamber temp, T_c:

- Energy used to dissociate instead of heat gas
- Want stable species at higher velocity (H2 at large velocity, not H at lower velocity)

Adiabatic combustion: (similar to our adiabatic flame temp. analysis)

$$C_p T_c = C_p T_i + \alpha\, q_c - \beta\, q_d \tag{9.2}$$

$T_i =$ initial temp
$\alpha =$ mass fraction of reactants burned completely
$\beta =$ mass fraction of dissociated products
$q_c =$ heat of combustion (total possible heat release by reactants)
$q_d =$ heat of dissociation (energy required to dissociate products)

If recombination, then - For example:

$$2\,H \longrightarrow H_2$$

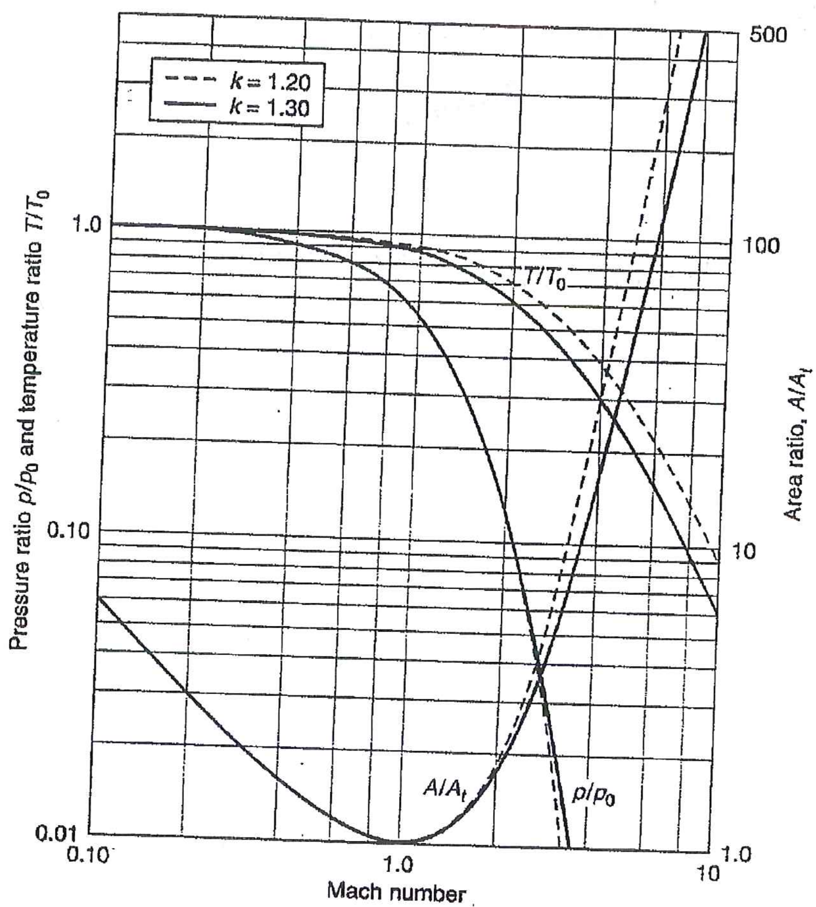
Heat released into flow, velocity increase and thrust increase.

3.2.2 Two extreme cases

(a) Equilibrium Flow

- Recombination as gas expands in nozzle, T decreases and H atoms recombine to form H2
- Amount of H and H2 exactly predicted by equilibrium calculations

Remember, T and p decrease as gas expands in nozzle. Our equilibrium composition calculations (see Ch. V D.) showed how equilibrium composition is dependent on T and p (higher T, more dissociation, higher p, less dissociation)



$u_e \cong \sqrt{2C_pT_c}$

$u_e \cong \sqrt{2C_p(T_c-T_e)}$

$u_e = \sqrt{\frac{2\gamma R}{(\gamma-1)MW} T_c \left[1-\left(\frac{p_e}{p_c}\right)^{\gamma/\gamma-1}\right]}$

Figure: Nozzle Location

(b) Frozen flow

- Mole fractions don't change
- Flow moves through, expands out nozzle too fast
- No time for composition to adjust

Isp (Equlibrium) > Isp (frozen)

So, as T drops through nozzle, dissociation should drop in equilibrium, but in reality, if gas moves too fast it never equilibrates.

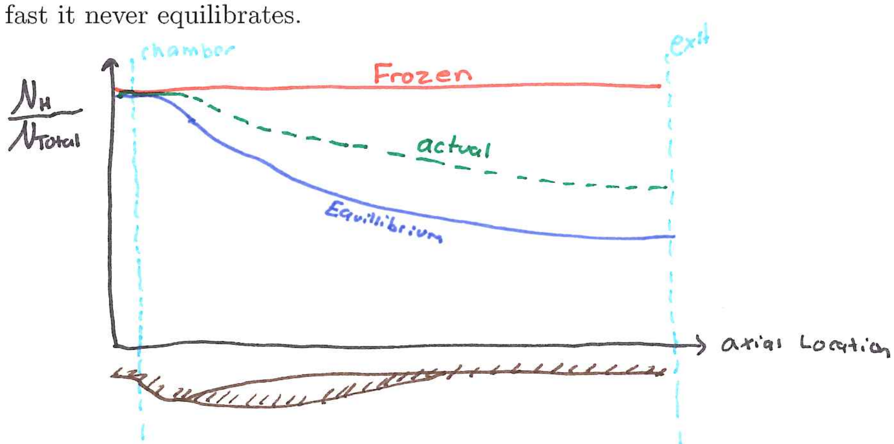


Figure: Frozen and Equilibrium Dissociation

In general, we can write:

$$\frac{\text{Chemical Reaction Time}}{\text{Residence Time}} = \begin{cases} \gg 1 & \text{Frozen} \\ \ll 1 & \text{Equillibrium} \end{cases} \tag{9.13}$$

The reaction rate is of the form:

$$ChemicalReactionTime \approx \frac{1}{\text{Reaction Rate } (k)} \approx A \exp\left(\frac{E}{RT}\right) \tag{9.14}$$

Then

$$\frac{A \exp\left(\frac{E}{RT}\right)}{\frac{\Delta Z}{u}} = \frac{\text{Chemical Reaction Time}}{\text{Residence Time}} = \begin{cases} \gg 1 & \text{Frozen} \\ \ll 1 & \text{Equilibrium} \end{cases} \tag{9.15}$$

Where

$$\begin{aligned} \Delta Z &= \text{Characteristic Length} \\ u &= \text{Flow Speed} \\ A &= \text{Constant} \\ E &= \text{Activation Energy} \\ k &= \text{Reaction Rate} \end{aligned}$$

Near inlet: T large, u small $\longrightarrow$ Equilibrium

Near Exit: T small, u large $\longrightarrow$ Frozen

- As flow accelerates and expands out nozzle, goes from being purely equilibrium to progressively more frozen.
- We can define a transition point, before which the flow is modeled as equilibrium and after which it is modeled as frozen. This is called the "quick freeze" point.

Summary:

Frozen flow conditions:

- Slow chemical reactions
- Short residence time
- Low density
- Too rapid change in p and T

Equilibrium favors large, high pressure performance motors

Frozen favors low pressure, small motors

3.3 Solid Propellants

Storable for a long time, dense propellant with reasonable performance

Pros

- High density
- High thrust and acceleration
- Storable
- Reliable
- Cheap
- Low maintenance
- Safe

Cons

- Lower Isp (<300 sec)
- Cannot be restarted or throttled
- Tends to pollute
- Difficult to make very large (brittle) - segments

3.3.1 Propellant Types

(Propellant = "grain")

(a) Homogeneous (typically "double base")

- Fuel and oxidizer contained in same molecule, Ex: Nitroglycerin-nitrocellulose with additives (called "double-base" propellants, very common)

TABLE 12.3 Typical Double-Base Propellant (JPN)

Material	Weight %	Purpose
Nitrocellulose (13.25%N)	51.40	Polymer
Nitroglycerin	42.93	Explosive plasticizer
Diethyl pthalate	3.20	Nonexplosive plasticizer
Ethyl centralite	1.00	Stabilizer
Potassium sulfate	1.20	Flash suppressor
Carbon black (added)	0.20	Opacifying agent
Candelilla wax (added)	0.07	Die lubricant

- Additives are often metals
- Small engines in air-air missiles
- Isp 150-200 sec

(b) Heterogeneous (typically "composite")

- Oxidizing crystals in organic plasticlike fuel binder
- Oxidizing crystal (commonly found in fertilizer)
  - Potassium perchlorate (KClO4)

  - Ammonia perchlorate ($NH4ClO4$)
  - Lithium perchlorate ($LiClO4$)
  - Potassium nitrate ($KNO3$)
- Binder (is the fuel, and also the glue that holds it all together)
  - Asphalt, rubber
  - Hydroxyl terminated polybutaliene (HTPB)
  - Copolymer of butadiene, acrylic acid, acrylonitride (PBAN)
- Typically heated to soften, malleable, mixed with oxidizing crystals, placed into a mold to cure.

(c) Metal Powder Additives

- Increase Isp and fuel density (density-specific impulse)
  - Aluminum $\longrightarrow$ SRBs Isp 260sec
  - Beryllium $\longrightarrow$ toxic Isp 285sec (Beryllium lung disease)
  - Boron $\longrightarrow$ combustion difficult Isp 290sec

Space Shuttle Solid Rocket Boosters

- 15% by mass aluminum
- 73% ammonium perchlorate
- 12% PBAN

In the exhaust, Al $\longrightarrow$ Al2O3 (alumina, solid ceramic particle), is 32% of the total mass flow

(d) Two-phase Flow

- Plume of solid rocket motor has high temperature gaseous exhaust containing aggregates of solid particles
- Metal additives agglomerate (coalesce) at burning surface into aggregates ($\approx 200\,\mu m$ size, which is huge compared to gas atoms $0.0001\,\mu m$ size)
- Accurate analysis must include both gas and solid propellant product exhaust (can be 1/3 of the mass flow!)

3.4 Solid Motor Analysis

Consider a generic solid rocket motor:

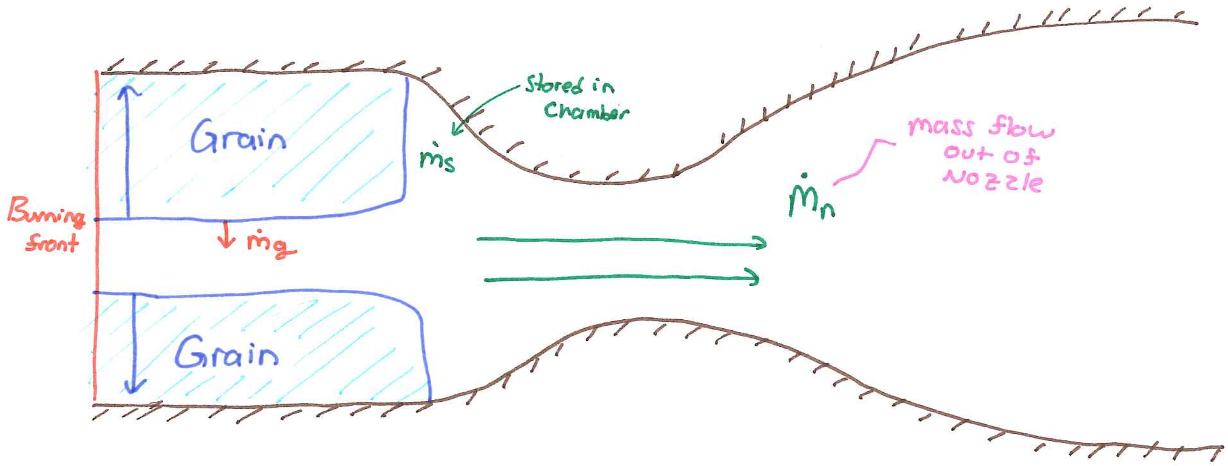


Figure: Generic Solid Rocket Motor

Rate of generation of gaseous propellant must equal the rate of consumption of the grain as it burns:

Rate of Generation of Gaseous Propellants

$$\dot{m}_g = \rho_b \, A_b \, r \tag{9.16}$$

Where

$$\begin{aligned} \rho_b &= \text{Solid Density} \\ A_b &= \text{Burning Area} \\ r &= \text{Recession Rate of Burning Front} \end{aligned}$$

The recession rate, r, can be written as:

Recession Rate

$$r = a \, P_o^n \tag{9.17}$$

Where

$$\begin{aligned} a &= \text{Constant (Empirical)} \\ n &= \text{Constant (Empirical)} \\ P_o &= \text{Combustion Chamber Pressure} \end{aligned}$$

As an approximation:

$$a = \frac{A}{T_1 - T_P} \tag{9.18}$$

Where: A and T_1 are both empirical constants. T_p = initial propellant temp

Now, recognize that the mass generated by combustion/burning of the grain can go two places (1) it can go out the nozzle, or (2) become stored in the combustion chamber (the chamber volume gets bigger as the grain burns, so it can hold more mass, $PV = MRT$, V goes up, M goes up, P&T are const).

Mathematically then:

$$\dot{m}_g = \dot{m}_s + \dot{m}_m \tag{9.19}$$

With (9.16) and (8.7),

$$\rho_b\, A_b\, r = \rho_o\, A_b\, r + \mathcal{V}_o \frac{\partial \rho_o}{\partial t} + \frac{P_o\, A^*}{\sqrt{R T_o}} \sqrt{\gamma \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}} \tag{9.20}$$

$$\rho_b\, A_b\, a\, P_o^n = \rho_o\, A_b\, a\, P_o^n + \frac{\mathcal{V}_o}{R T_o} \frac{\partial P_o}{\partial t} + \frac{P_o\, A^*}{\sqrt{R T_o}} \sqrt{\gamma \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}} \tag{9.21}$$

Then

$$\frac{\mathcal{V}_o}{R T_o} \frac{\mathrm{d} P_o}{\mathrm{d}t} = A_b\, a\, P_o^n\, (\rho_b - \rho_o) - \sqrt{\frac{\gamma}{R T_o} \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}\; A^*\, P_o \tag{9.22}$$

This is how the pressure in the chamber changes in time based on the burning area, propellant characteristics and chamber temperature.

If dp_o/dt small (steady state) compared to RHS, then set LHS = 0, then the equilibrium pressure is:

$$P_o = \left[\frac{A_b}{A^*} \frac{a\,(\rho_b - \rho_o)}{\sqrt{\frac{\gamma}{R T_o} \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}}\right]^{\frac{1}{1-n}} \tag{9.23}$$

Also, we can generally assume $\rho_b \gg \rho_o$, the solid grain density is much larger than the gaseous product density, so $\rho_b - \rho_o \approx \rho_b$

Engine thrust is proportional to the chamber pressure (remember the thrust coefficient (8.12)). And now we see the chamber pressure is proportional to the burning area, A_b. So this is a means of changing the solid rocket motor thrust, change the burning area. The burning area will change with time depending on the grain geometry.

This gives rise to "progressive", "neutral", or "regressive" burning profiles.

- "Progressive" - A_b and thrust increase with time
- "Neutral" - A_b and thrust constant with time
- "Regressive" - A_b and thrust decrease with time

Ex: Progressive - Area of inner diameter as time increases, the area is increasing and so the burning area increases with time therefore the thrust increases with time.

Section 3: Air Breathing Propulsion

AE435

Spring 2018

1 Aircraft Jet Engines (Airbreathing Propulsion)

Contents

1.1 Introduction

- Vehicle carries fuel, uses atmosphere air as oxidizer
- Does not carry oxidizer

$$\dot{m}_{air} \cong 50\dot{m}_{fuel} \tag{10.1}$$

- Propellers typical for low-speeds, piston engine drives propeller at low powers
- Gas turbines used for large propellers and at higher speeds (less mass per power output)
- $M > 0.5$ propellers inefficient and noisy, switch to turbojet or turbofan
- Our analysis concentrates on air-breathing engines using a gas turbine cycle (No piston or propeller analysis, see internal combustion engines course ME403 and applied aerodynamics course AE416)

1.2 Thrust and Efficiency

1.2.1 Thrust

See our previous analysis, eqn. (3.13) and associated handout in Ch. III.

$$F = \dot{m}_e \, u_e - \dot{m}_{\text{air}} \, u + (P_e - P_a) \, A_e \tag{10.2}$$

Where

$$\begin{aligned} e &= \text{exit} \\ \text{air} &= \text{inlet air intake} \\ u &= \text{inlet/flight velocity} \end{aligned}$$

Recall that mass flow...

$$\dot{m}_e = \dot{m}_a + \dot{m}_f$$

Where

$$\begin{aligned} a &= \text{air} \\ f &= \text{fuel} \end{aligned}$$

Which gives us

Fuel-Air Ratio

$$f = \frac{\dot{m}_f}{\dot{m}_a} \tag{10.3}$$

Finally we have

Thrust

$$F = \dot{m}_a \big[(1 + f) \, u_e - u \big] + (P_e - P_a) \, A_e \tag{10.4}$$

1.2.2 Efficiency

(a) **Propulsion Efficiency**
Propulsion Efficiency

$$\eta_P = \frac{F\,u}{\dot{m}_a\left[(1+f)\,\frac{u_e^2}{2} - \frac{u^2}{2}\right]} \tag{10.5}$$

This is energy conversion efficiency. How efficiently the Cycle Power (which creates high KE exhaust) is converted into Propulsion Power (thrust force doing work on vehicle).

- **Numerator:**
  Thrust power or propulsion power is: $F\,u$
  This is the actual power applied to the vehicle.
  - **Propulsion Work** - $F\,dx$ (force x distance = work, work being done by the thrust force moving the vehicle a distance dx)
  - **Propulsion Power** - $F\frac{\mathrm{d}x}{\mathrm{d}t} = F\,u$ (rate at which propulsive work is being done)
- **Denominator:**
  - Cycle power, the power or net rate at which energy is being put into the propellant stream. The rate of production of KE by the propellant.
  - Cycle Power = Propellant stream Power Out - Power In

In general, $f \ll 1$. If pressure thrust is small, then

$$F \approx \dot{m}_a\,(u_e - u)$$

Then:

$$\eta_P = \frac{(u_e - u)\,u}{\frac{u_e^2}{2} - \frac{u^2}{2}} = \frac{2\left(\frac{u}{u_e}\right)}{1 + \frac{u}{u_e}} \tag{10.6}$$

(b) **Thermal Efficiency**

$$\eta_{th} = \frac{\text{Cycle Power}}{\text{Total Fuel Power}} = \frac{\dot{m}_a\left[(1+f)\,\frac{u_e^2}{2} - \frac{u^2}{2}\right]}{\dot{m}_f\,Q_R} \tag{10.7}$$

Where

$$Q_R = \text{Heating value of fuel} \qquad \left[\frac{kJ}{kg}\right]$$

$$\eta_{th} = \frac{(1+f)\,\frac{u_e^2}{2} - \frac{u^2}{2}}{f\,Q_R} \tag{10.8}$$

In some cases the Output Cycle Power is shaft power, P_s, then:

$$\eta_{th} = \frac{P_s}{\dot{m}\, Q_R} \tag{10.9}$$

(c) **Overall Efficiency**

$$\eta_o = \eta_{th}\, \eta_p = \frac{F\, u}{\dot{m}_f\, Q_R} \tag{10.10}$$

If we assume $f \ll 1$ and pressure thrust is small, then with (10.6):

$$\eta_o \cong 2\, \eta_{th} \left(\frac{u/u_e}{1 + u/u_e} \right) \tag{10.11}$$

(d) **Additional Considerations**

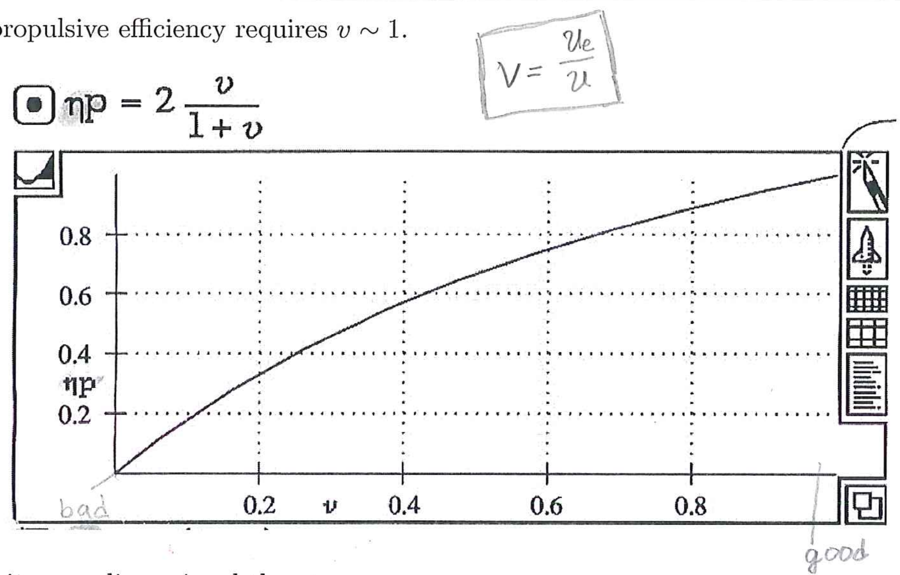
Let $u/u_e = v$, this is flight velocity divided by exhaust exit velocity.

Propulsive Efficiency:

Then (10.6) is:

$$\eta_p = \frac{2\, v}{1 + v}$$

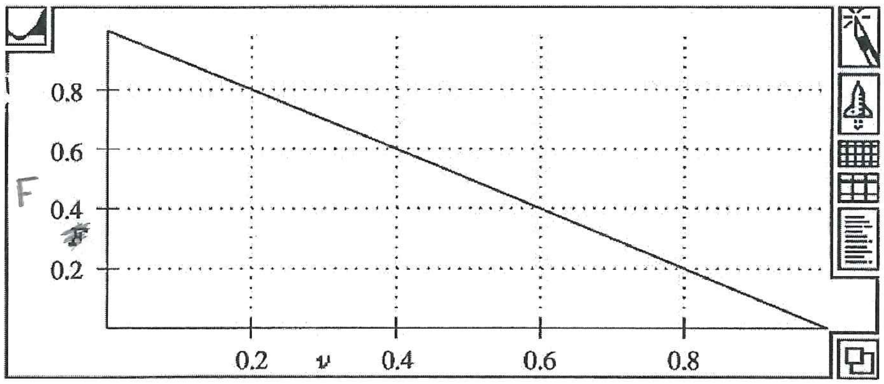
High propulsive efficiency requires $v \sim 1$.



Can write non-dimensional thrust as:

$$F \approx \dot{m}_a\, (u_e - u) \quad \rightarrow \quad \frac{F}{\dot{m}_a\, u_e} \cong 1 - v \tag{10.12}$$

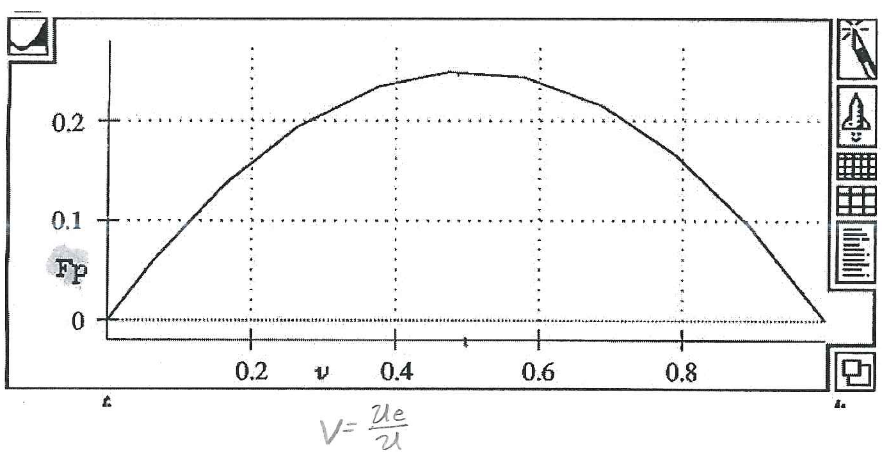
But high thrust requires, $v \sim 0$.



Can write non-dimensional thrust/propulsive power as:

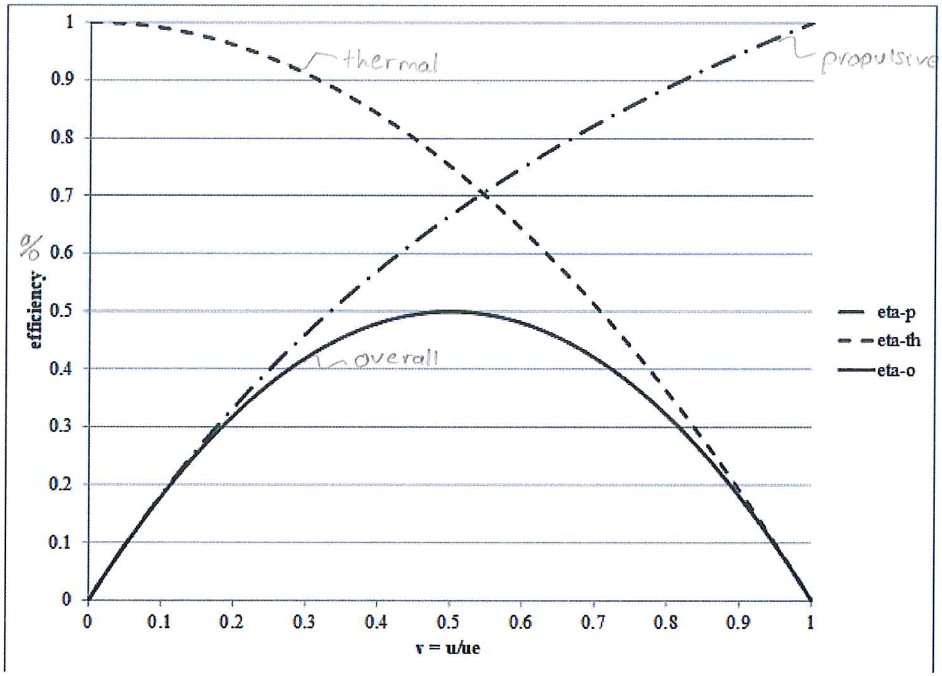
$$F \approx \dot{m}_a \, (u_e - u) \, u \qquad \rightarrow \qquad \frac{F\,u}{\dot{m}_a \, u_e^2} = v\,(1 - v)$$

High thrust power requires $v = 0.5$



Efficiencies:

We see that max overall efficiency is when $v \sim 0.5$ or $(u_e \sim 2\,u)$
(remember we neglected pressure thrust and assumed $f \ll 1$)
Keep this general result in mind as we move to the next section!



Finally, we can write the overall efficiency in terms of an ideal exhaust velocity. "Ideal exhaust velocity" is the max possible exhaust velocity if all the chemical bond energy was converted to KE of the fluid.

$$\dot{m}_f\, Q_R = (\dot{m}_f + \dot{m}_a)\,\frac{u_{\text{ideal}}^2}{2}$$

Then:

$$\eta_o \cong \frac{\dot{m}_a\,(u_e - u)\,u}{(\dot{m}_f + \dot{m}_a)\,\frac{u_{\text{ideal}}^2}{2}}$$

If $\dot{m}_f \ll \dot{m}_a$

$$\eta_o \approx \frac{2\,(u_e - u)\,u}{u_{\text{ideal}}^2} = 2\,v\,(1-v)\left(\frac{u_e}{u_{\text{ideal}}}\right)^2 \qquad (10.13)$$

What value of v will maximize overall efficiency? You already know from the figure above. But we can show by taking derivative and setting equal to zero.

$$\frac{d\eta_o}{dv} = 2(1-2v)\left(\frac{u_e}{u_{\text{ideal}}}\right)^2 = 0$$

And we find max overall efficiency when $v = 1/2$ or $(u_e \sim 2\,u)$.

Substituting this result back in, we get:

$$\eta_{o,\max} = \frac{1}{2}\left(\frac{u_e}{u_{\text{ideal}}}\right)^2 \tag{10.14}$$

Note that $u_e < u_{\text{ideal}}$ (can't get more energy out than you put in!), so what does this say the max possible efficiency will be?

In general, propulsive efficiency is quite high (60-80%, KE of expelled propellant pretty efficiently converted to vehicle power), but thermal efficiency is much lower (20-40%, difficult to convert chemical bond energy to cycle power), such that overall efficiencies of 25-30% are common. We will see:

(a) To get higher thermal efficiency, need higher compressor pressure ratio.

(b) To get higher compressor pressure ratio, need higher burner temperature, which implies need higher temp materials.

1. **Importance of Efficiency**

Unlike chemical rockets where these energy conversion efficiencies are not the most important parameters (the mass economy, that is, specific impulse is more important), they are important in gas turbine engines.

Like Isp (or c) is important for determining rocket delta-V.

Overall efficiency, η_o, is important for determining aircraft range, R.

Consider steady-level flight.

$$F = D + L = W$$

Using (10.10),

$$F\,u = \eta_o\,\dot{m}_f\,Q_R = D\,u$$

Which can be rewritten as:

$$\frac{\eta_o\,\dot{m}_f\,Q_R}{W} = \frac{D}{L}\,u$$

Recognizing that the fuel mass flow rate:

$$\dot{m}_f = -\frac{1}{g}\frac{\mathrm{d}W}{\mathrm{d}t}$$

Rearranging and integrating we get:

$$-\frac{\eta_o Q_R}{g}\frac{\mathrm{d}W}{W} = \frac{D}{L}u\mathrm{d}t = \frac{\mathrm{d}R}{L/D}$$

Or

$$R = \eta_o \frac{Q_R}{g}\frac{L}{d}\ln\left(\frac{W_i}{W_f}\right) \quad (10.14a)$$

Higher efficiency gives rise to longer range.

Again using (10.10) into (10.14a):

$$R = \frac{F\,u}{\dot{m}_f\,Q_R}\frac{Q_R}{g}\frac{L}{d}\ln\left(\frac{W_i}{W_f}\right)$$

And letting $u = M\,a$ and defining a new parameter, the Thrust Specific Fuel Consumption (TSFC)

$$\mathrm{TSFC} = \frac{\dot{m}_f}{F}$$

TSFC is an engine performance parameter, how much thrust you get for input fuel rate.

We see the range can now be written:

$$R = \left(M\frac{L}{D}\right)\frac{1}{\mathrm{TSFC}}\frac{a}{g}\ln\left(\frac{W_i}{W_f}\right) \quad (10.14b)$$

So to get long range, want large $M\frac{L}{D}$, this is called the "range factor" and is related to the aerodynamics of the vehicle, it's the lift-to-drag at Mach, $\#M$.

Also want low/small TSFC (get lots of thrust without using a lot of fuel!)

Finally, note that TSFC is the specific impulse of the airbreathing propulsion system. That is:

$$I_{SP} = \frac{F}{\dot{m}\,g} = \frac{1}{\mathrm{TSFC}\,g} \quad (10.14c)$$

1.3 Air Breathing Engines

1.3.1 Turbojet

- Air inducted through inlet and enters compressor.
- Compressor is turbomachinery, consisting of rotor and stator blade pairs (one pair is one stage). Rotors are attached to hub, which is attached to a centerline shaft. Shaft spinning at high RPM 10,000's RPM.
- Compressed air enter combustor where fuel is added, mixed, and combusted with the air.
- High temperature high pressure combustion gas enters turbine.
- Turbine is turbomachinery, consisting of nozzle/stator and rotor blade pairs connected to hub and shaft. High temperature/pressure gas impinges on rotor blades, resulting force does work on hub and shaft. That work is transferred upstream to drive the compressor.
- Gas expands through turbine then out nozzle and exhausted from engine at high speed.

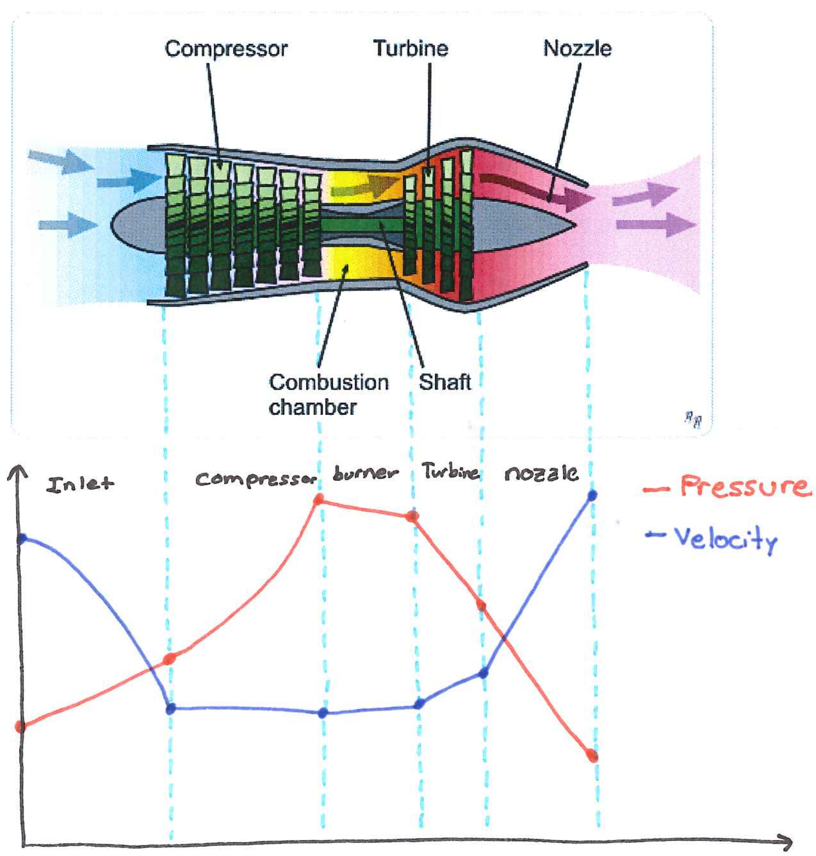


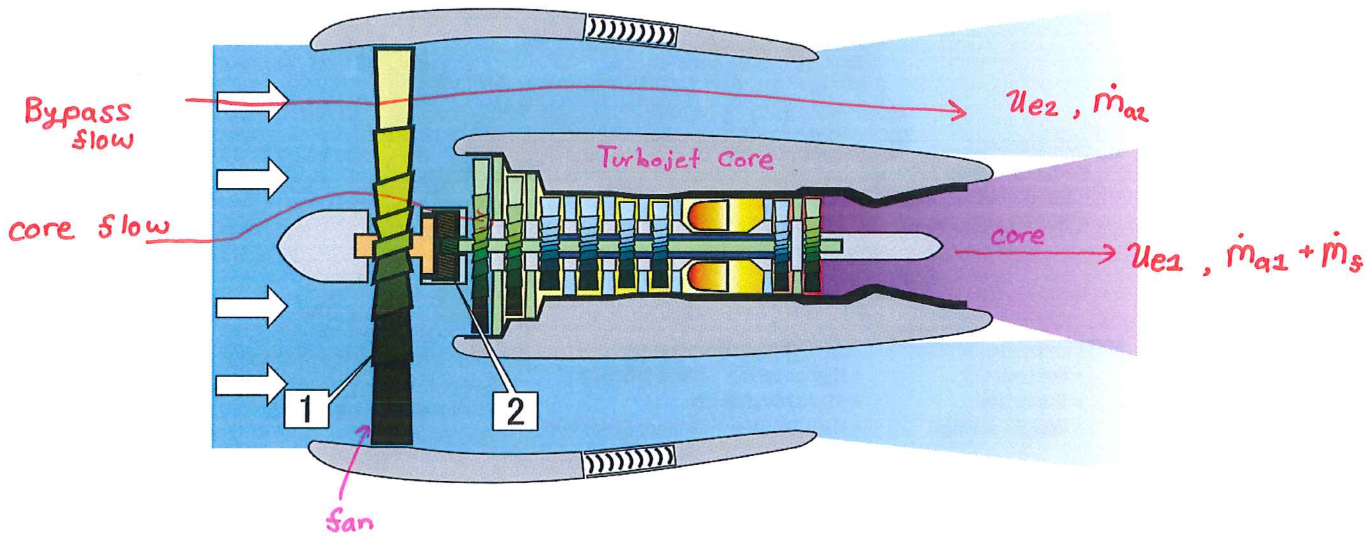
Figure: Velocity and Pressure profile through Turbojet:

Turbojets are used on older aircraft:

- 707, DC-8, DC-9, 737 (very loud)
- $u_e \sim 1200$ [m/s] $\sim 5u$ ($u \sim 240$ [m/s] ~ 540 mph, Mach .85) Wasteful.
- Remember, optimum was $u_e \sim 2\,u$

1.3.2 Turbofan

Like the turbojet, but has a bypass. This lowers the effective exhaust velocity closer to 2u (better efficiency). Turbojet core, surrounded by a bypass stream of air.



$$\beta = \text{Bypass Ratio} = \frac{\dot m_{a,2}}{\dot m_{a,1}}$$

Exhaust velocity:

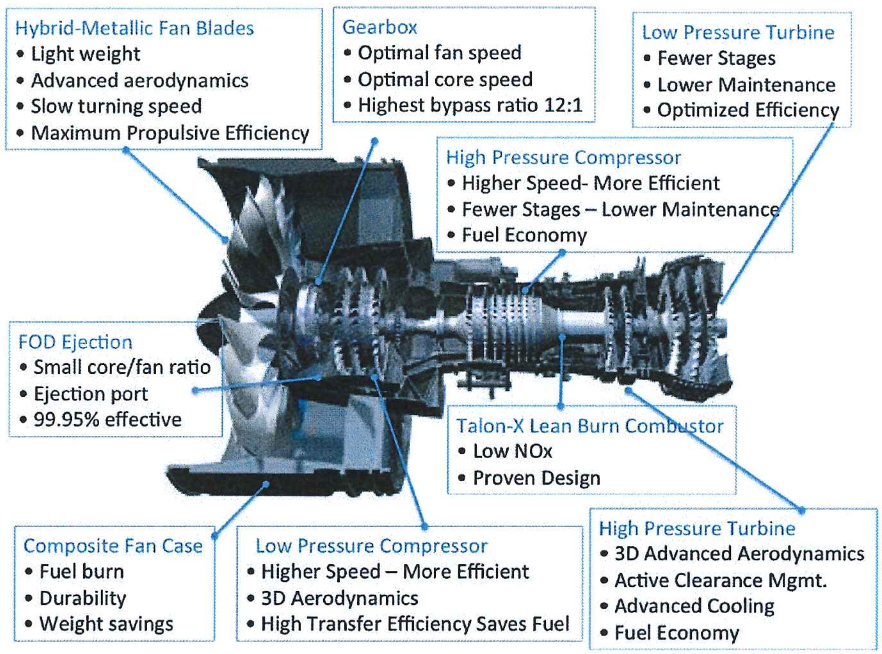
$$u_e = \left(\frac{\dot m_{a,1}}{\dot m_{a,1} + \dot m_{a,2}}\right) u_{e,1} + \left(\frac{\dot m_{a,2}}{\dot m_{a,1} + \dot m_{a,2}}\right) u_{e,2}$$

Larger $\dot m_a$ results in more F, while lower u_e yields better efficiency.

$$\begin{aligned} u_{e,1} &\sim 600 - 1200 \quad [m/s] \\ u_{e,1} &\sim 300 - 400 \quad [m/s] \\ u_e &\sim 500 - 600 \quad [m/s] \end{aligned}$$

- Commercial aircraft: DC-10 $\beta \sim 4$ C5 A/B $\beta \sim 8$
- Military aircraft: F15/F16 $\beta \sim 0.36$

PW1000G - probably most advanced gas turbine engine in the World, ultra-high bypass ratio ($\beta \sim 12 - 14$)



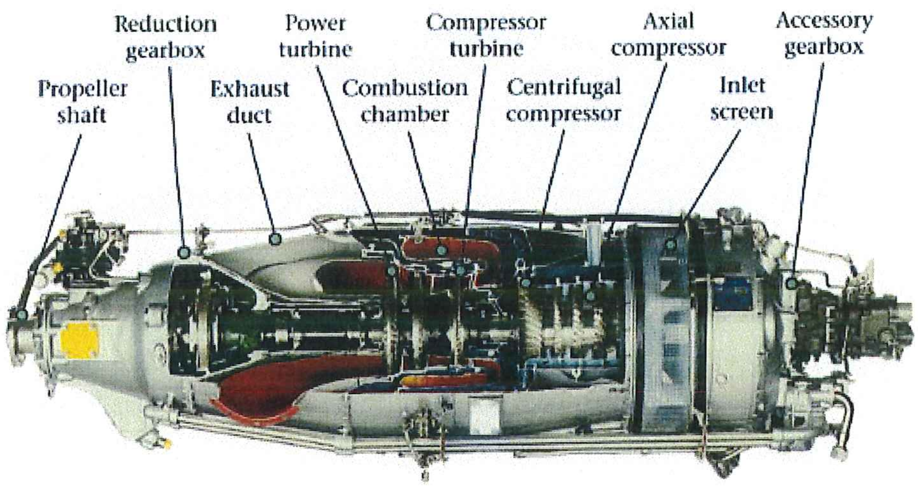
PRATT & WHITNEY ENGINE CHARACTERISTICS

Feature v Model >	V2533-A5	PW4098	PW6122A	GP7270	PW1524G	F119-PW-100	F135-PW-100
Engine Type	High Bypass Turbofan	High Bypass Turbofan	High Bypass Turbofan	Very High Bypass Turbofan	Ultra High Bypass Turbofan	Augmented Turbofan	Augmented Turbofan
Number of fan / LPC / HPC stages	1 / 4 / 10	1 / 7 / 11	1 / 4 / 6	1 / 5 / 9	1 / Gear / 3 / 8	3 / 0 / 6	3 / 0 / 6
Number of HP / LP turbine stages	2 / 5	2 / 7	1 / 3	2 / 6	2 / 3	1 / 1	1 / 2
Combustor type	Annular	Annular	Annular	Annular	Annular	Annular	Annular
Maximum Std Day thrust at sea level (lbs)	31,600	99,040	22,100	74,735	23,300	35,000 class	43,000 class
Specific fuel consumption at takeoff power	0.355	0.358	0.383				
Overall pressure ratio at max. power	33.4	42.8	26.6	36.1			28
Bypass Ratio	4.5	5.8	5.0	8.7			
Fan Pressure Ratio	1.8	1.8	1.7	1.6			
Diameter (inches)	64	120	62	124	79		46
Length, flange to flange (inches)	126	192	108	187	120		220
Weight, bare engine (lbs)	5,300	16,260	5,041	14,798			
Application for this model	Airbus A321-200	Boeing 777-300	Airbus A318-100	Airbus A380-800	Bombardier CS100, CS300	Lockheed F-22	Lockheed F-35A (CTOL)
Other Applications for the engine family	Airbus A319, A320, Boeing MD-90	Boeing 777-200			Embraer E190, E195 Second Generation		Lockheed F-35B (STOVL), F-35C (CTOL Carrier Variant)
Takeoff TT4 (degC)	1550	1748	1582				
Cruise TT4 (degC)	1139	1197	1235				
Takeoff EGT (degC)	670	675	760				
Max Continuous EGT (degC)	610	617	727				

1.3.3 TurboProp (Unducted Turbofan)

- Best for short take-off and low-speed flight
- Propeller inefficient at high speeds
- Unducted fan better at higher speeds, but not as high a pressure ratio due to unducted operation.

P&W CanadaPT6



Note the reverse flow path, propeller not shown.

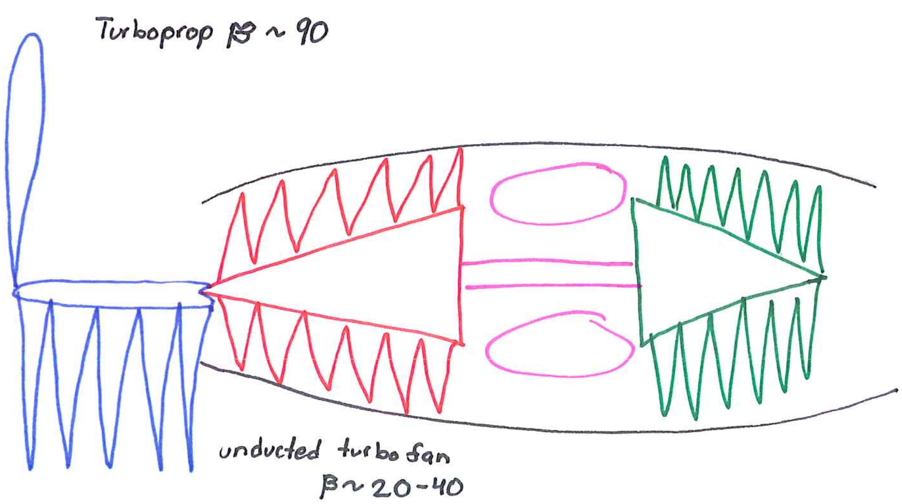
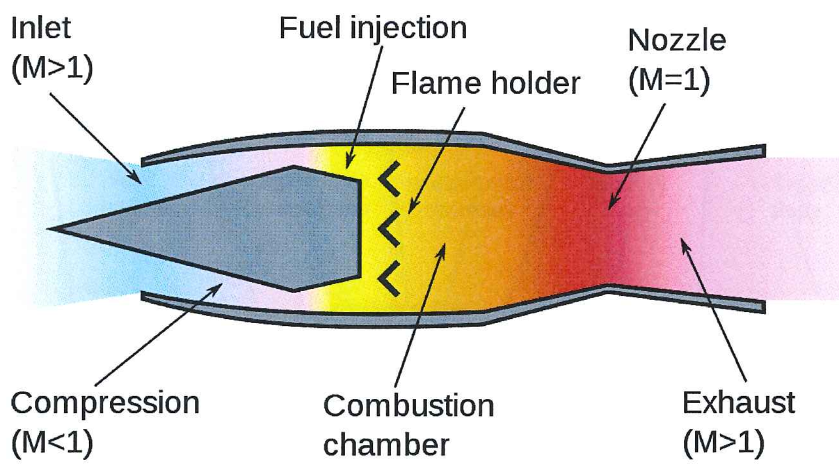


Figure: TurboProp Inside

1.3.4 Ramjet

- Good for $1 < M < 6$ or 7
- At $M \sim 7$, $T \sim 2200K$, temp too large
- Gasdynamic compression (no turbomachinery, no compressor, no turbine)



1.3.5 Scramjet

Supersonic combustion ramjet

Challenges: internal flow $M > 1$

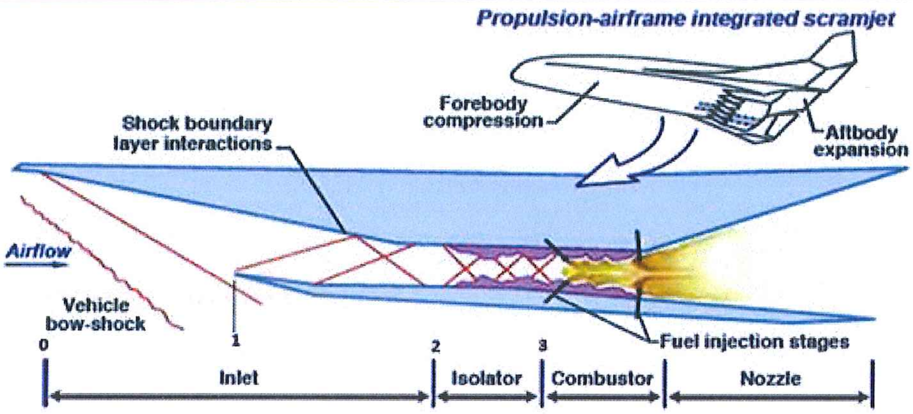
For example: $M \sim 25$, then $M_{\text{internal}} \sim 8$, so T low enough

At $M \sim 8$, $u = 7200$ m/s

If engine is 10 m long, that's long, $\tau_{\text{combust}} \sim \frac{10\,[m]}{7200\,[m/s]} \sim 1.5[ms]$

This is not long enough to injet, mix, and combust fuel.

Scramjets generally have clever internal shock and gasdynamic structure geometry to enable combustion.



Why do we need these different engines? When to use turboprop vs turbofan vs turbojet?

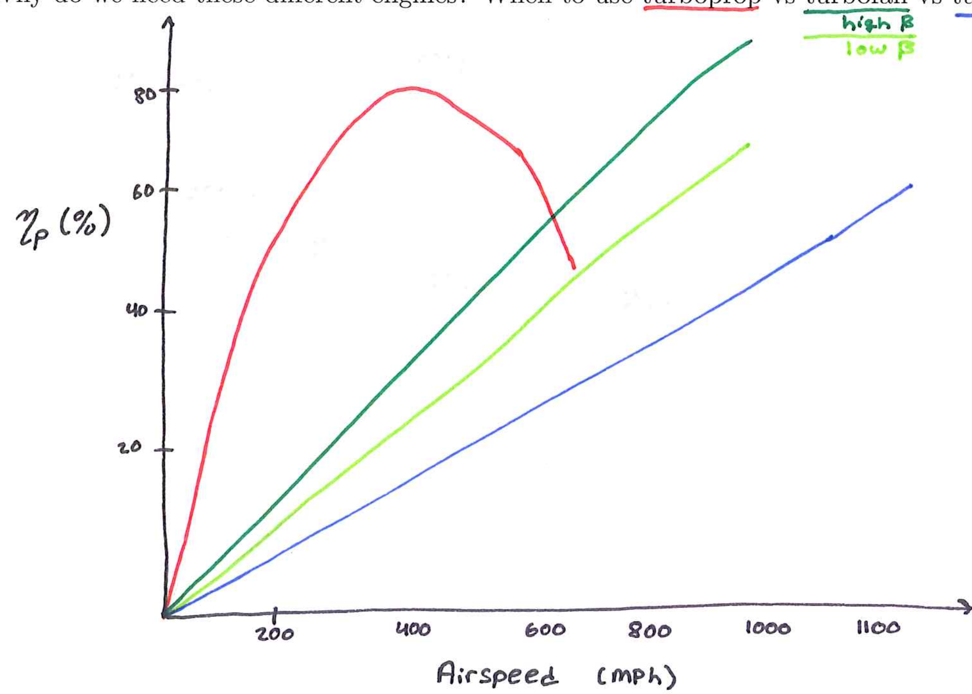


Figure: Propulsion Efficiency V. Airspeed

$$F = \dot{m}_a \underbrace{\left[(1+f)\,u_e - u\right]}_{f \ll 1} + \underbrace{(P_e - P_a)\,A_e}_{\text{Assume Small}}$$

$$F \cong \dot{m}_a\, u \left(\underbrace{\frac{u_e}{u}}_{\sim 2 \text{ for max } \eta_o} -1 \right)$$

$$F_{\max\, \eta_o} \cong \dot{m}_a\, u$$

To maintain F at lower u (flight speed) need to actuate/energize a larger $\dot{m}_a$ amount of air.

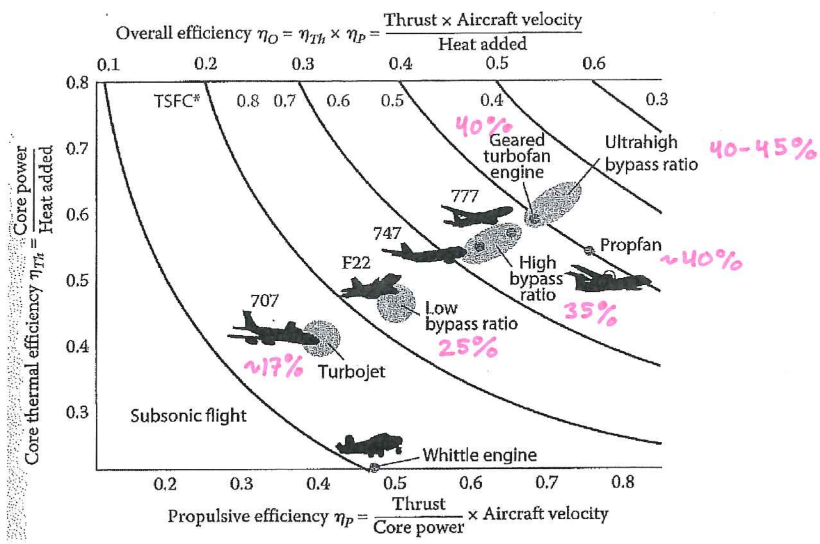
Turbojet ⟶ Turbofan ⟶ Unducted Turbofan ⟶ Turboprop

———— Increasing Actuated Mass $\dot{m}_a$ ————→

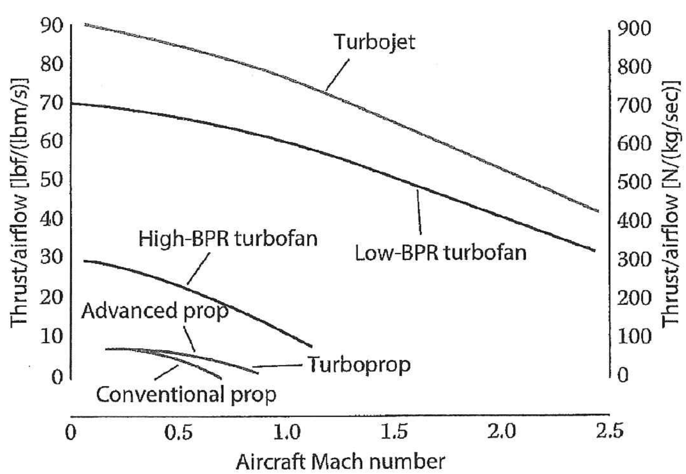
———— Decreasing Exhaust Velocity u_e ————→

such that $u_e \sim 2u$

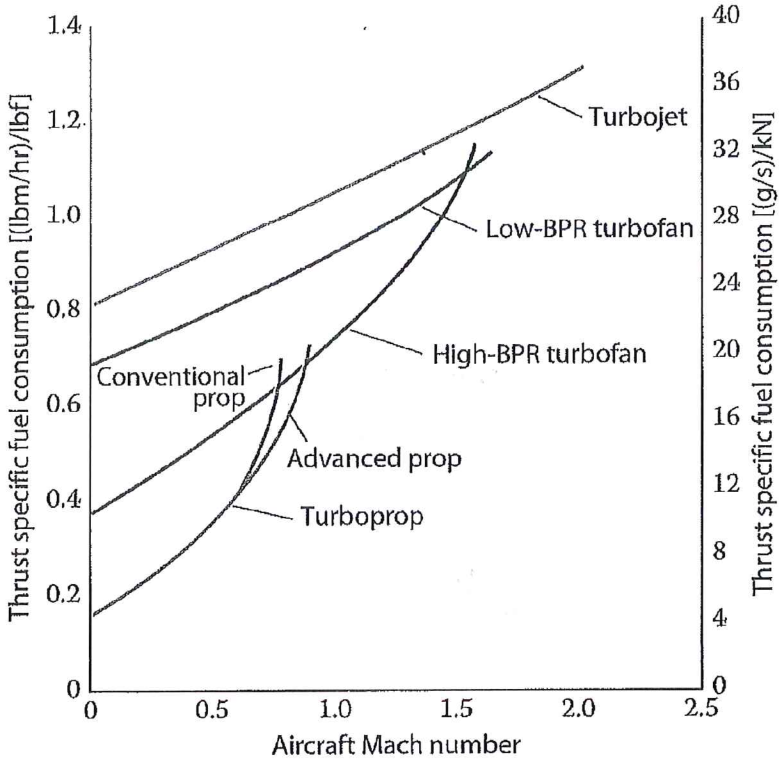
Efficiencies of different engine types



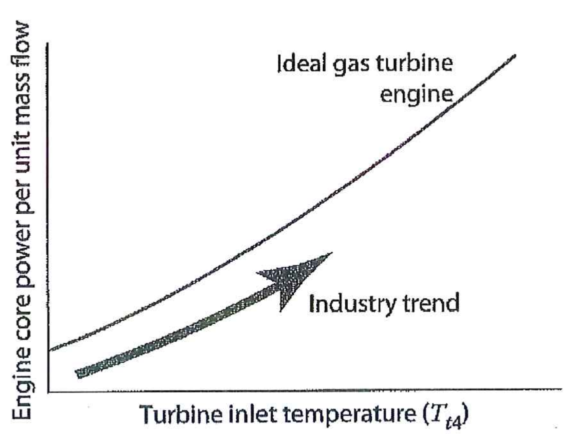
Specific thrust ($F/\dot{m}_a$) vs Mach number



TSFC vs Mach #



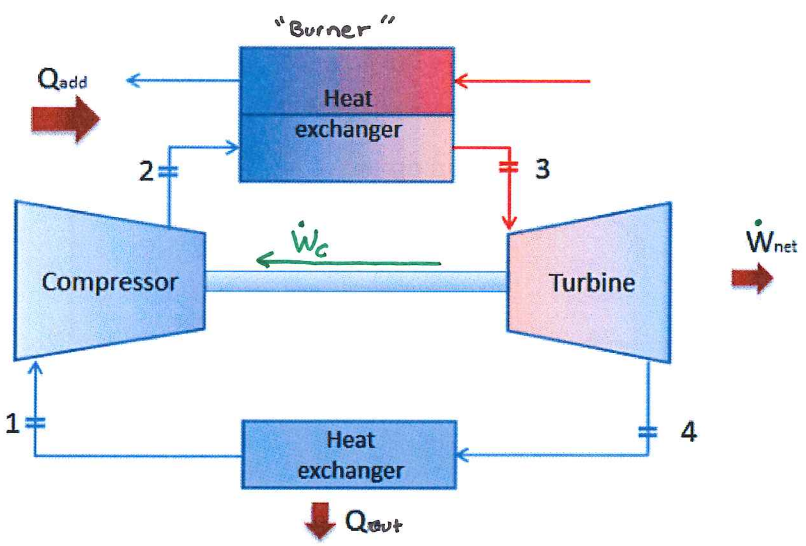
Industry trend is towards very high bypass ratio engines (for commercial aircraft). To get high efficiency requires high compressor pressure ratio, which in turn requires high combustor exit (turbine inlet) temperatures. Hence the industry trend is toward higher turbine inlet temperature, which requires development of new high temperature turbine materials. Efficiencies approaching 40-45% are becoming possible.

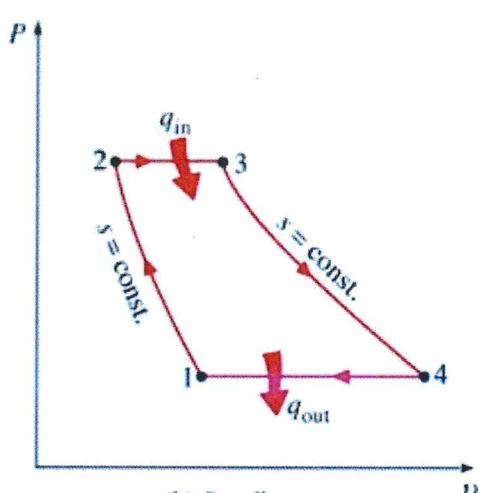


1.4 Brayton Cycle

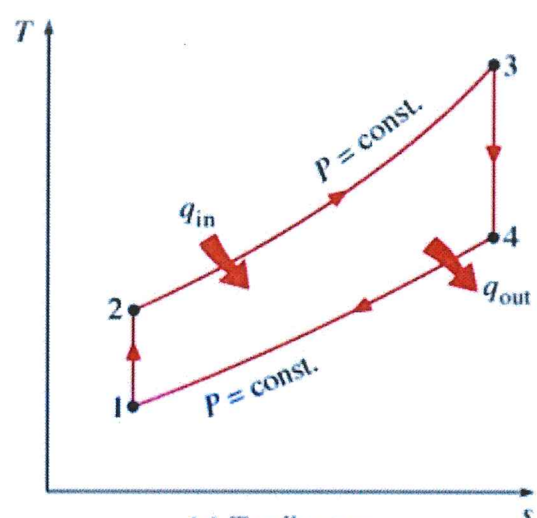
- Brayton cycle is a thermodynamic cycle used to model gas turbines.
- Four main processes: (closed cycle)
  1. Isentropic compression
  2. Constant pressure heat addition
  3. Isentropic expansion
  4. Constant pressure heat rejection (open cycle eliminates step 4)

Schematic:





(b) P- υdiagram



(a) T-s diagram

Assuming calorically perfect gas (C_p const)

$$\begin{aligned} \text{Compressor} \quad & \dot{W}_c = \dot{m}\, C_p\, (T_2 - T_1) \\ \text{Turbine} \quad & \dot{W}_t = \dot{m}\, C_p\, (T_3 - T_4) \\ \text{Burner} \quad & \dot{Q}_{\text{in}} = \dot{m}\, C_p\, (T_3 - T_2) \\ \text{Heat Rejector} \quad & \dot{Q}_{\text{out}} = \dot{m}\, C_p\, (T_4 - T_1) \end{aligned} \tag{10.15}$$

$$\text{Net } \dot{W}_{\text{out}} = \dot{W}_t - \dot{W}_c = \dot{Q}_{\text{in}} - \dot{Q}_{\text{out}} = \dot{m}\, C_p \left[(T_3 - T_4) - (T_2 - T_1) \right] \tag{10.16}$$

1.4.1 Thermal efficiency

$$\eta_{th} = \frac{\dot{W}_{\text{out}}}{\dot{Q}_{\text{in}}} = \frac{\dot{Q}_{\text{in}} - \dot{Q}_{\text{out}}}{\dot{Q}_{\text{in}}} = 1 - \frac{\dot{Q}_{\text{out}}}{\dot{Q}_{\text{in}}} \tag{10.17}$$

$$\eta_{th} = 1 - \frac{C_p\, (T_4 - T_1)}{C_p\, (T_3 - T_2)} \tag{10.18}$$

This can be rewritten using the fact that $P_2 = P_3$ and $P_1 = P_4$,

$$\left(\frac{P_2}{P_1}\right)^{\frac{\gamma-1}{\gamma}} = \left(\frac{P_3}{P_4}\right)^{\frac{\gamma-1}{\gamma}} = \frac{T_2}{T_1} = \frac{T_3}{T_4}$$

$$\frac{T_3}{T_2} = \frac{T_4}{T_1} \quad \longrightarrow \quad \eta_{th} = 1 - \frac{T_1}{T_2} = 1 - \left(\frac{P_1}{P_2}\right)^{\frac{\gamma-1}{\gamma}} \tag{10.19}$$

Let,

Compressor Pressure Ratio

$$r_p = \frac{P_2}{P_1}$$

Such that...

$$\eta_{th} = 1 - \frac{1}{r_p^{\frac{\gamma-1}{\gamma}}} \tag{10.20}$$

Work output, as before:

$$W_{out} = C_P\,(T_3 - T_4) - C_P\,(T_2 - T_1)$$

$$= C_P\,T_1 \left[\left(\frac{T_3}{T_1} - \frac{T_4}{T_1} \right) - \left(\frac{T_2}{T_1} - 1 \right) \right]$$

With (10.18a)

$$\frac{W_{out}}{C_P\,T_1} = \eta_{th} \left[\frac{T_3 - T_2}{T_1} \right] \tag{10.21}$$

1.4.2 Backwork Ratio

The fraction of the compressor power to the turbine power. We generally want this to be small so that the turbine extracts as much work as possible (W_t large) and we want the compressor to not use much of the work generated by the compressor (W_c small)

$$\frac{W_c}{W_t} = \frac{\text{Compressor Work}}{\text{Turbine Work}}$$

We want this ratio to be small.

T2 not T1 ??

$$\frac{W_c}{W_t} = \frac{T_2 - T_1}{T_3 - T_4} = \frac{T_1}{T_3}\, r_p^{\frac{\gamma-1}{\gamma}} \tag{10.22}$$

1 – comp in
3 – Burner out

Example: For an ideal Brayton cycle:

$$T_1 = 300\,K \qquad T_3 = 1800\,K \qquad r_p = 10 \qquad \gamma = 1.4$$

Solution:

$$\frac{W_c}{W_t} = 32\,\%$$

32% of the work from the turbine has to go back to the compressor.

In reality:

1. Non-isentropic engine
2. Decreased work by turbine
3. Compressor requires more input work from turbine

Real Systems:

$$\frac{W_C}{W_t} > 0.7$$

1.4.3 Fixed Temperature Ratio Brayton Cycle - Optimization

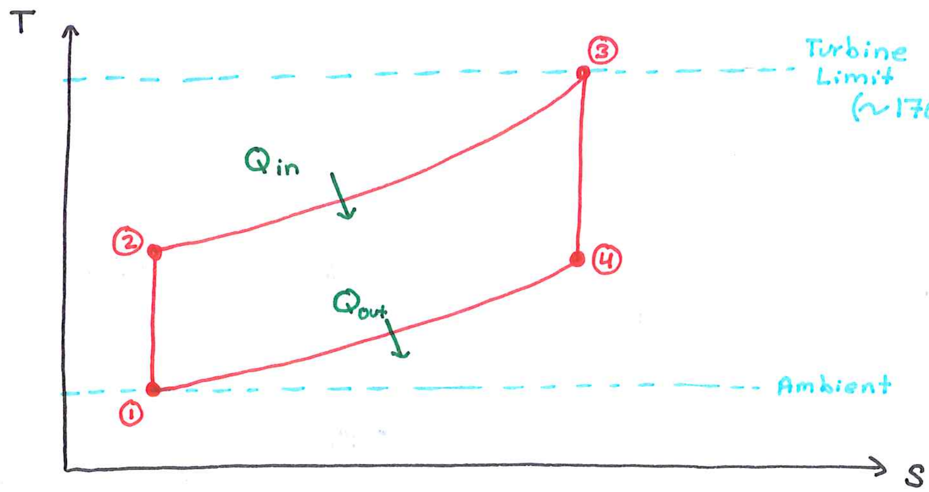


Figure: T-s Diagram for Ideal Brayton Cycle

We will be working with the fixed temperature ratio Brayton cycle. This means that T_1 is going to be fixed based on the ambient environment. T_3 will also be fixed, it is the max temperature the system can withstand.

Our Brayton cycle must operate between those dashed lines in the figure above.

Recognize that this is just 10.21 again.

$$\frac{W_{out}}{C_P T_1} = \eta_{th}\left[\frac{T_3 - T_2}{T_1}\right] = \left[1 - \frac{T_1}{T_2}\right]\left[\frac{T_3}{T_1} - \frac{T_2}{T_1}\right]$$

Recognize 1-2 is isentropic therefore:

$$\frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^{\frac{\gamma-1}{\gamma}}$$

Such that:

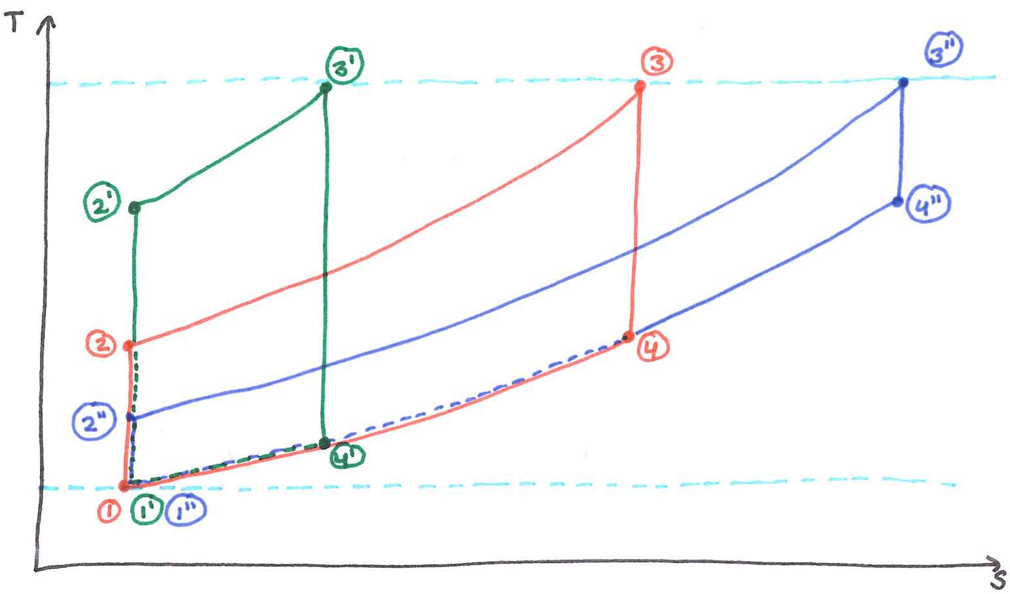
$$\frac{W_{out}}{C_P T_1} = \left[1 - \left(\frac{P_2}{P_1}\right)^{\frac{\gamma-1}{\gamma}}\right]\left[\frac{T_3}{T_1} - \left(\frac{P_2}{P_1}\right)^{\frac{\gamma-1}{\gamma}}\right]$$

How do we get the most output work when this temperature ratio is fixed. What Pressure ratio will give us the most output work.

So, the net output Work W_{out} is a function of $P_2/P_1 = r_p$, compressor pressure ratio, and the temperature ratio T_3/T_1.

This ratio is set by the ambient (intake air) temperature and the turbine max possible temperature. So we ask the question, is there an optimum pressure ratio (r_p) that gives the most W_{out} for a given temperature ratio?

Consider two different Brayton Cycles on T-S diagram:



1'-2'-3'-4'

- Large compression ratio ∴ Large η_{th}
- But... $T_{2'} \sim T_{3'}$ so

$$\frac{W_{out}}{C_P T_1} = \frac{\eta_{th}}{T_1}\left[T_3 - T_2\right] \approx 0$$

∴ Not much output work

- Lots of compression.

1-2''-3''-4''

- Small compression ∴ small η_{th}
- $T_{3''} >> T_{2''}$
- Again W_{out} is small
- Lots of fuel.

Figure: T-s Diagram for Brayton Cycle

Define:

$$\frac{T_3}{T_1} = \theta \qquad \left(\frac{P_2}{P_1}\right)^{\frac{\gamma-1}{\gamma}} = \pi$$

$$\frac{W_{out}}{C_P T_1} = \left(1 - \frac{1}{\pi}\right)(\theta - \pi) \tag{10.23}$$

θ is a constant and π is the variable that can change here.

$$\frac{\mathrm{d}}{\mathrm{d}\pi}\left(\frac{W_{out}}{C_P T_1}\right) = 0 = \frac{\mathrm{d}}{\mathrm{d}\pi}(\theta - \pi - \frac{\theta}{\pi} + 1) \rightarrow$$

We find that:

$$\theta = \pi^2 \qquad \text{or} \qquad \pi = \sqrt{\theta} \tag{10.24}$$

We are trying to find the maxumum work output which leads us to:

$$\left(\frac{P_2}{P_1}\right)_{optimal} = \left(\frac{T_3}{T_1}\right)^{\frac{\gamma}{2(\gamma-1)}} \tag{10.25}$$

Such that the maximum optimal work output is...

Maximum Optimal Work Output given a Fixed Temperature Brayton Cycle

$$\left.\frac{W_{out}}{C_P T_1}\right|_{optimal} = \left[1 - \left(\frac{T_3}{T_1}\right)^{-\frac{1}{2}}\right]\left[\frac{T_3}{T_1} - \left(\frac{T_3}{T_1}\right)^{\frac{1}{2}}\right] \tag{10.26}$$

What about our maximum pressure ratio? How do we get the maximum pressure across the compressor? What is the maximum π?

We get the maximum pressure if all the turbine work goes into the compressor. In this case the back work ratio is 1.

When the back work ratio is 1, the output work is 0 because there is no work leaving the system.

$$\frac{W_c}{W_t} = 1 = \frac{T_1}{T_3}\left(\frac{P_2}{P_1}\right)^{\frac{\gamma-1}{\gamma}} \quad \rightarrow \quad \frac{P_2}{P_1} = \left(\frac{T_3}{T_1}\right)^{\frac{\gamma}{\gamma-1}} \tag{10.27}$$

With (10.25) this means that

$$\left(\frac{P_2}{P_1}\right)_{max} = \left(\frac{P_2}{P_1}\right)^2_{optimized} \tag{10.28}$$

The maximum pressure ratio is just the square of the optimal pressure ratio.

Example:

$$T_1 = 300\,K \qquad T_3 = 1800\,K \qquad \gamma = 1.4 \qquad \left.\frac{P_2}{P_1}\right|_{max} = 529 \qquad \left.\frac{P_2}{P_1}\right|_{optimized} = 23$$

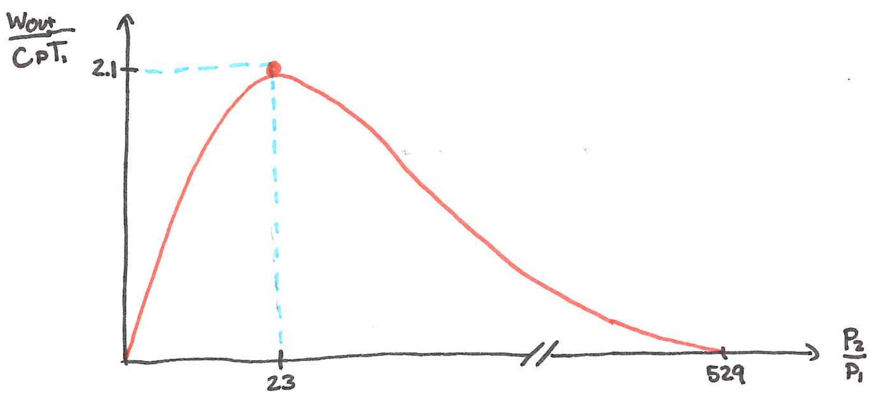


Figure: Output Work V. Pressure Ratio

Section 3: Air Breathing Propulsion

AE435

Spring 2018

1 Ideal Cycle Analysis

Contents

1.1 Introduction

- Study thermodynamic change of working fluid as it flows through engine
- Parametric cycle analysis vs. Engine Performance Analysis
  - Parametric cycle analysis - performance of engine with different design choice (compressor ratio, bypass ratio, etc.)
  - Engine performance analysis - performance of a given engine at all applicable flight conditions
- We are doing parametric cycle analysis, want to determine how engine performance (F thrust and TSFC thrust specific fuel consumption) affected by design choices (compressor ratio, bypass ratio), design limitations (burner temp), and flight environment (M#, Pa, Ta)
- We will look at the ramjet, turbojet, and turbofan
- Common metric for comparing airbreathing engines is: Thrust Specific Fuel Consumption (TSFC)

1.2 Ideal Ramjet

Simplest (conceptually) of all airbreathing engines

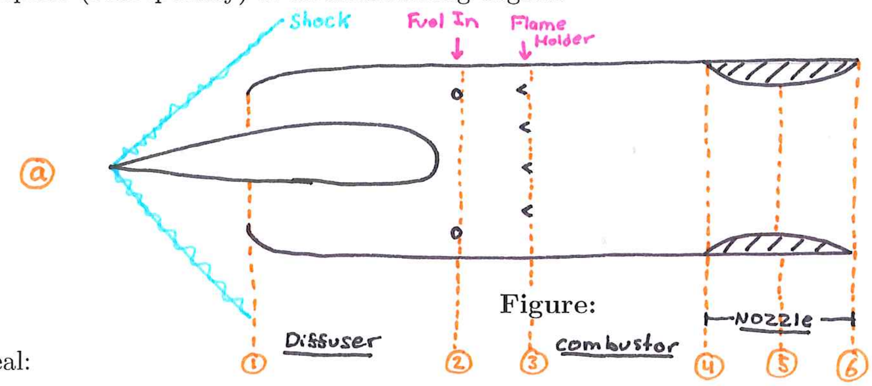


Figure:

Ideal:

- Compression & expansion are isentropic processes
- Constant pressure combustion/heat addition
- Performance highest thermodynamics allows

Real:

- Losses due to shocks, mixing and wall friction in diffuser
- Review of Rayleigh-Fanno flow → total pressure and static pressure drop during combustion
- Nozzle will have friction and heat transfer

Our model:

(a) → (o2)	Isentropic compression to stagnation state (2)
(o2) → (o4)	Constant pressure heating and mass addition up to T_{o4}
(o4) → (6)	Isentropic expansion to ambient state

On a T-S diagram:

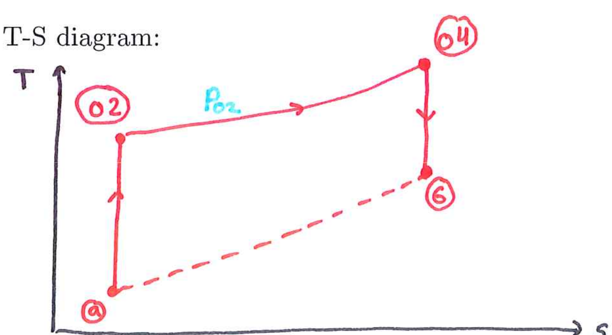


Figure:

Ideal thrust is then:

$$F = \dot{m}_a \left[(1+f) u_e - u \right] \tag{11.1}$$

Since $P_e = P_a = P_6$

Since isentropic compression and expansion, and constant pressure heat addition $\rightarrow P_o =$ constant everywhere!

If also assume R and γ constant everywhere too, (also $P_{o,a} = P_{o,6}$) then:

$$\frac{P_{oa}}{P_a} = \left(1 + \frac{\gamma - 1}{2} M_a^2\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{P_{o,6}}{P_6} = \left(1 + \frac{\gamma - 1}{2} M_6^2\right)^{\frac{\gamma}{\gamma-1}}$$

Where now...

$$\frac{P_{oa}}{P_a} = \frac{P_{o,6}}{P_6} \qquad M_e = M_a = M_6 \tag{11.2}$$

So $u_e = u_6 = \frac{a_e}{a_a} u = \sqrt{\frac{T_e}{T_a}} u$

$$\frac{T_e}{T_a} = \frac{T_{o,6}}{T_{o,a}} \quad \text{and} \quad T_{o,4} = T_{o,6} \quad \rightarrow \quad u_e = \sqrt{\frac{T_{o,4}}{T_{o,a}}} u \tag{11.3}$$

Energy Equation for the combustor yields:

$$\dot{m}_a h_{o2} + \dot{m}_f Q_R = (\dot{m}_a + \dot{m}_f) h_{o4}$$

$$(1+f) h_{o4} = h_{o2} + f Q_R \tag{11.4}$$

Where

$$f = \text{Fuel-Air Ratio}$$
$$Q_R = \text{Fuel Heating Value [kJ / kg]}$$

If constant C_p = constant, then

$$f = \frac{(T_{o4}/T_{oa}) - 1}{\frac{Q_R}{C_P T_{oa}} - (T_{o4}/T_{oa})} \tag{11.5}$$

Using (11.1) and (11.3) yields:

$$\frac{F}{\dot{m}_a} = M\sqrt{\gamma R T_a}\left[(1+f)\sqrt{\frac{T_{o,4}}{T_a}}\left(1+\frac{\gamma-1}{2}M^2\right)^{-1/2} - 1\right] \tag{11.6}$$

where f is equation (11.5)

Then TSFC is:

$$TSFC = \frac{\dot{m}_a}{F} = \frac{f}{\frac{F}{\dot{m}_a}} \tag{11.7}$$

Even though this is an IDEAL ramjet model, the trends are very similar for real ramjets, as the figures below show.

IDEAL RAMJET FIGURE

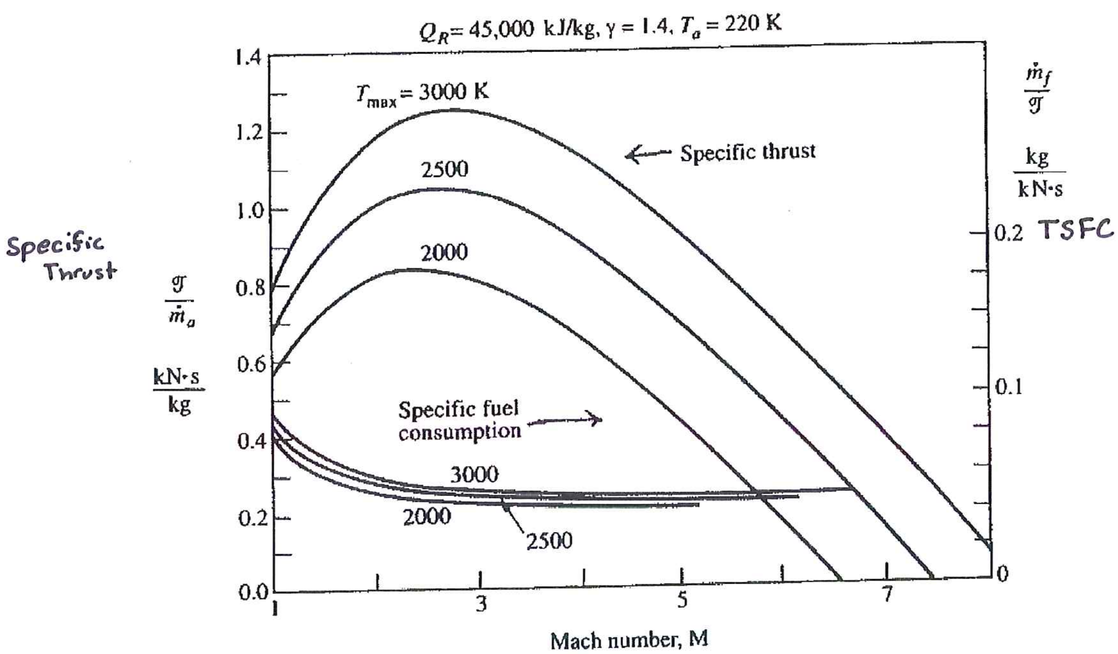


FIGURE 5.9 Ideal ramjet thrust and fuel consumption.

IDEAL Thrust and Efficiencies

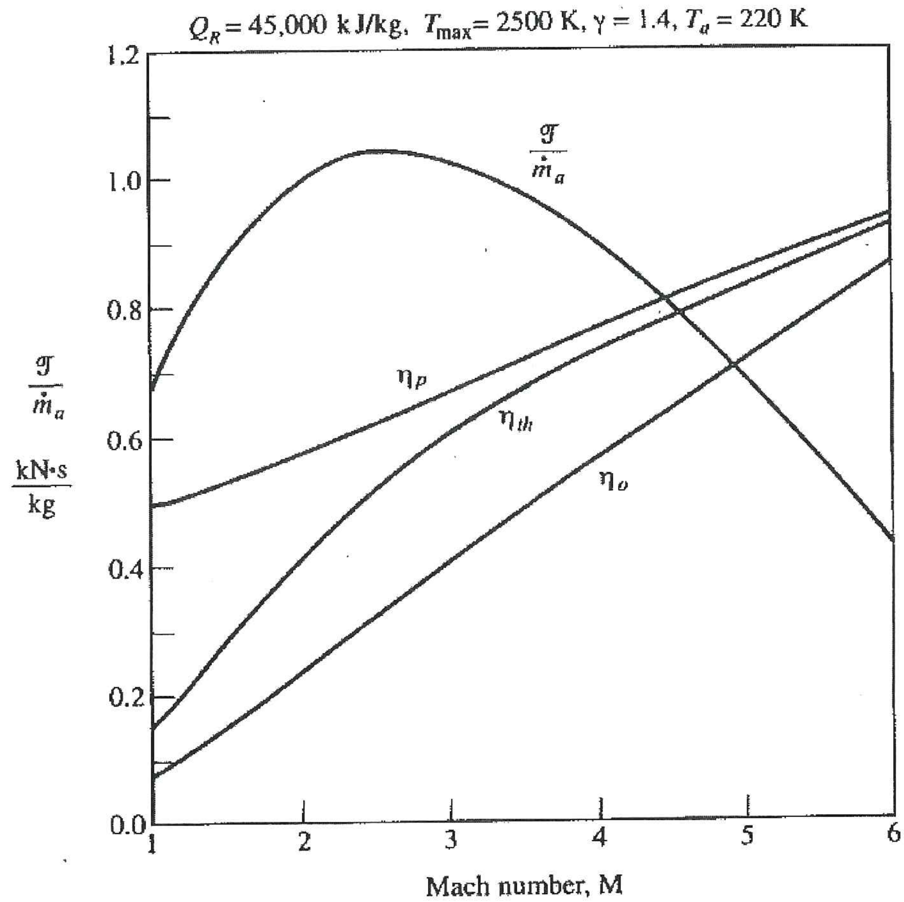


FIGURE 5.10 Ideal ramjet thrust and efficiencies.

- Highest specific thrust at $M \sim 2.6 - 2.7$, from Eqn. (11.6).
- TSFC essentially plateaus after $M \sim 2.5$. TSFC is an initially decreasing function with Mach number, and then plateaus. This is because of eqn. (11.7). The denominator of (11.7) is the specific thrust (11.6), which is an increasing, then decreasing function per the figure. The numerator of (11.7), f, is also a function of Mach # per eqn. (11.5). We can re- write Eqn. (11.5) in terms of Mach # as:

$$f = (11.5) = fnc(M) = \frac{\frac{T_{o4}}{T_a} - \left(1 + \frac{\gamma-1}{2} M^2\right)}{\frac{Q_R}{C_P T_{oa}} - \frac{T_{o4}}{T_a}}$$

This shows that f is a decreasing function with M (which is expected, a higher Mach # flow has a higher total temperature T_{oa}, so need less fuel to bring the temperature up to T_{o4}). So, at low Mach #'s, specific thrust (denominator, Eqn. 11.7) is increasing, while fuel-air ratio (numerator) is decreasing, overall effect is fast decrease in $TSFC$. At higher Mach #'s, specific thrust is decreasing and fuel-air ratio is decreasing, and they are decreasing at the same rate, so no change in $TSFC$, $TSFC$ plateaus.

- Note that the $TSFC$ curves just stop at some Mach #. From our Eqn. above, $f = fnc(M)$, at some Mach #, f will become negative, an impossible result, hence the curve stops at that Mach #. This plot doesn't go to that $f = 0$ condition, but stops at about $f = 0.003$ ($M = 6.2$ for $T_{o4} = 2500\,K$)
- $M > 3$ could provide much greater range (higher overall efficiency). Overall efficiency increases sharply, while the specific thrust declines sharply, with increase M. Common result that conditions for minimum fuel consumption (or max range) are quite different from those for which engine size perp unit thrust is minimum.

$M = \text{low}$; $\frac{F}{\dot{m}_a}\uparrow$, f (numerator) $\downarrow$, TSFC $\downarrow$

$M = \text{high}$; $\frac{F}{\dot{m}_a}\downarrow$, $f\downarrow$, TSFC = constant (plateus)

REAL RAMJET (r_d, r_b, r_n, are diffuser, burner, nozzle total pressure ratios)

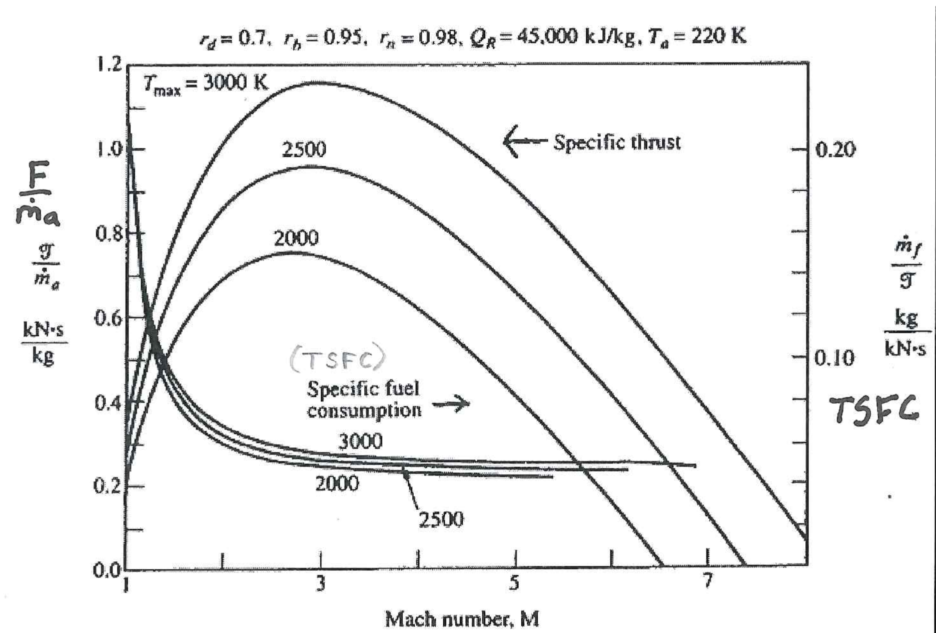


FIGURE 5.12 Ramjet thrust and fuel consumption.

Effects of aerodynamic losses.

What if compression, burning, and nozzle not isentropic/constant pressure?

Revised T-S diagram for REAL processes:

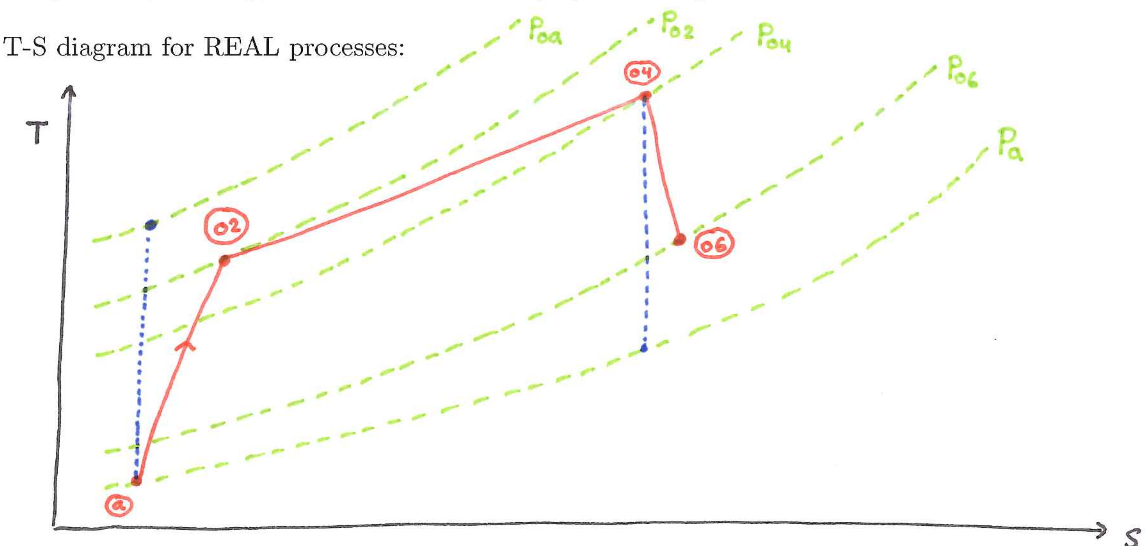


Figure:

Notes:

Start at same condition (a), but

- Now (02) higher S and lower T, compression was not adiabatic (a little heat lost), and not reversible (so P_o decrease)
- (02) to (04), not constant pressure, total pressure decreases, entropy increases.
- (04) to (6), expansion not back to ambient pressure P_a (not matched nozzle), also not isentropic, entropy increases.

Performance of the diffuser (inlet), combustor, and nozzle can be quantified using the total pressure ratio (remember, total pressure loss is related to entropy increases, see Eqn. 2.26 and 2.27).

Diffuser loss

$$\tau_d = \frac{P_{o2}}{P_{oa}} \tag{11.8}$$

Combustor loss

$$\tau_c = \frac{P_{o4}}{P_{o2}} \tag{11.9}$$

Nozzle loss

$$\tau_n = \frac{P_{o6}}{P_{o4}} \tag{11.10}$$

In an "ideal" ramjet, these are all = 1.

The total ramjet pressure ratio is then:

$$\frac{P_{o6}}{P_{oa}} = \tau_d \tau_c \tau_n \tag{11.11}$$

Now, including the aerodynamic losses, and $P_6 = P_e \neq P_a$

Then,

$$M_e^2 = \frac{2}{\gamma - 1}\left[\left(1 + \frac{\gamma - 1}{2} M^2\right)\left(\tau_d \tau_c \tau_n \frac{P_a}{P_e}\right)^{\frac{\gamma-1}{\gamma}} - 1\right] \tag{11.12}$$

And

$$u_e = M_e \sqrt{\frac{\gamma R T_{o4}}{1 + \frac{\gamma - 1}{2} M_e^2}} \qquad (11.13)$$

These irreversibilities have no effect on T_o, so f remains same, eqn. (11.5).

BUT, since $P_e \neq P_a$, now:

$$F = \dot{m}_a \left[(1+f)u_e - u\right] + \frac{1}{\dot{m}_a}\left(P_e - P_a\right) A_e \qquad (11.14)$$

Or, using (11.12) and (11.13), then (11.14) becomes:

$$\frac{F}{\dot{m}_a} = (1+f)\sqrt{\frac{\gamma R T_{o4}\,(\theta - 1)}{(\gamma - 1)\,\theta}} - M\sqrt{\gamma R T_a} + \frac{P_e A_e}{\dot{m}_a}\left(1 - \frac{P_a}{P_e}\right) \qquad (11.15)$$

where

$$\theta = \left(1 + \frac{\gamma - 1}{2} M^2\right)\left(\tau_d \, \tau_c \, \tau_n \frac{P_a}{P_e}\right)^{\frac{\gamma - 1}{\gamma}}$$

$$TSFC = \frac{f}{\frac{F}{\dot{m}_a}}$$

which is same, but now use (11.15) instead of (11.6)

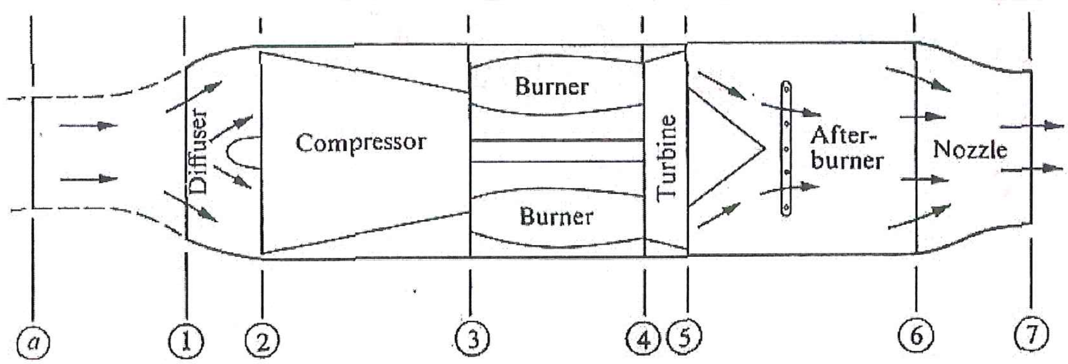
Assuming $\tau_d = 0.7$ $\quad \tau_c = 0.95$ $\quad \tau_n = 0.98$

See Figure 5.12. There is a less than 10% reduction in max specific thrust and TSFC (see figures above). However!, these losses are dependent on M#, not constant with Mach #. For example, the shock(s) in the diffuser inlet change as M# changes, causing the τ_d to change. Therefore to maintain optimum operation, it becomes desirable to adjust the inlet geometry (e.g., SR-71 Blackbird).

More on inlets, combustors and nozzles in our component analysis (Ch. XII.)

1.3 Turbojet

Schematic:



Processes:

- (a) → (1) - far upstream, flight conditions, air brought to inlet with some acceleration or deceleration of the flow
- (1) → (2) - air velocity decreased as air carried to compressor inlet
- (2) → (3) - air compressed in compressor
- (3) → (4) - air heated by mixing and burning with fuel
- (4) → (5) - air expanded through turbine to obtain power to drive compressor
- (5) → (6) - air may or may not be heated again by afterburner, more fuel added and burned
- (6) → (7) - air acceleration and exhausted through nozzle.

IDEAL Case:

1. Isentropic processes
2. Perfect gas
3. Constant pressure heating
4. Negligible velocities (2) → (6)

Resulting T-S diagram(s):

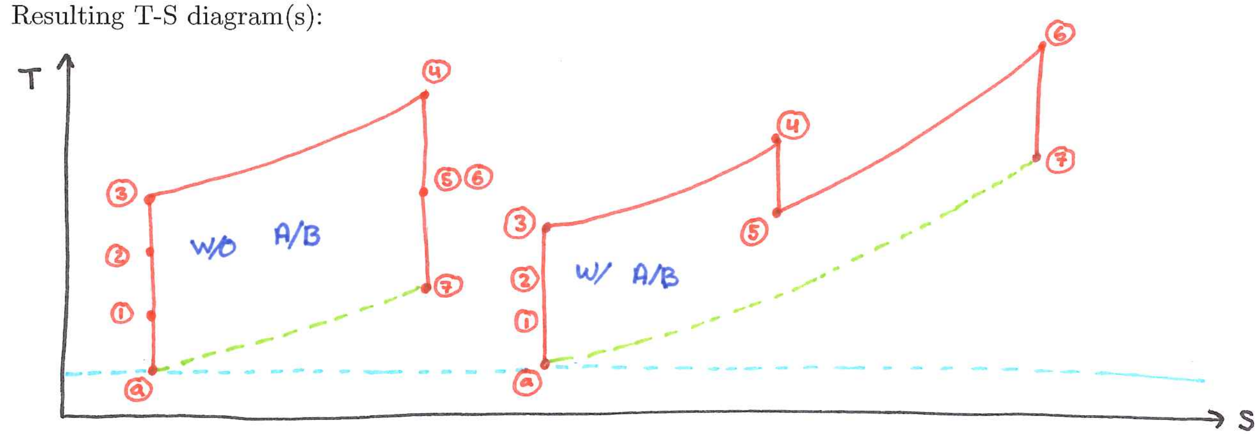


Figure:

REAL Case:

- ⓐ → ① isentropic
- ① → ② - friction, ds
- ② → ③ - non-isentropic compression
- ③ → ④ - not constant pressure combustion
- ④ → ⑤ - non-isentropic expansion
- ⑤ → ⑥ - loss in stagnation pressure due to heating and friction
- ⑥ → ⑦ - close to isentropic expansion

Real T-S diagram:

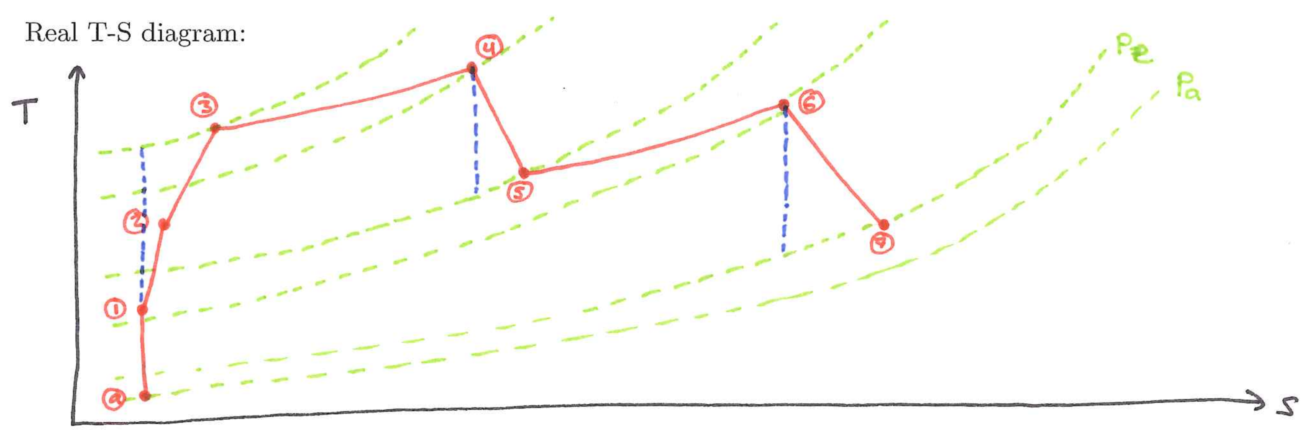


Figure:

Figure of Merit we are interested in, same as ramjet:

1. Specific thrust $F/\dot{m}_a$
2. Thrust specific fuel consumption $TSFC = \dot{m}_a/F = f/F/\dot{m}_a$

Again, ideal case $P_e = P_7 = P_a$, so

$$F = \dot{m}_a\left[(1+f)u_e - u\right]$$

Want large specific thrust since $\dot{m}_a \propto$ Engine Size (Weight)

Want small TSFC since $(TSFC)^{-1} \propto$ Aircraft Range

Now, step through each engine component:

Inlet/Diffuser ①→②

①

$$M = \frac{u}{\sqrt{\gamma R T_a}}$$

Adiabatic therefore

$$T_{oa} = T_{o2} = T_a\,(1 + \frac{\gamma - 1}{2}M^2) \tag{11.16}$$

Adiabatic Efficiency

$$\eta_d = \frac{h_{o2,s} - h_a}{h_{o2} - h_a} = \frac{\text{Isentropic Process}}{\text{Real Process}} \tag{11.17}$$

Typically the adiabatic efficiency ranges from $0.7 < \eta_d < 0.9$. We could also use the total pressure ratio, like we did for the Ramjet to get...

$$\frac{P_{o2}}{P_a} = \left[1 + \eta_d\left(\frac{T_{o2}}{T_a} - 1\right)\right]^{\frac{\gamma}{\gamma-1}} \tag{11.18}$$

Compressor Outlet (3)

$$r_c \text{ specified} \equiv \frac{P_{o3}}{P_{o2}}$$

Then,

$$P_{o3} = r_c \, P_{o2} \tag{11.19}$$

Compressor Efficiency

$$\eta_c = \frac{h_{o3,s} - h_{o2}}{h_{o3} - h_{o2}} \tag{11.20}$$

Typically this ranges from $0.8 < \eta_c < 0.9$

then we find that

(3) outlet
(2) comp in

$$\frac{T_{o3}}{T_{o2}} = \left[1 + \frac{1}{\eta_c}\left(r_c^{\frac{\gamma-1}{\gamma}} - 1\right)\right] \tag{11.21}$$

disconnect from Eqn 10.22

Burner ((3) → (4))

$$f = \frac{\frac{T_{o4}}{T_{o3}} - 1}{\frac{Q_R}{C_P T_{o3}} - \frac{T_{o4}}{T_{o3}}} \tag{11.22}$$

Typically $P_{o,4} = P_{o,3}$ since constant pressure heat addition or do Rayleigh/Fanno Flow to get $P_{o,4}$

Turbine ④ → ⑤

Power to Turbine = Power to Compressor.

Assuming all turbine power goes to the compressor...

$$\dot{W}_t = \dot{m}_t \left(h_{o,4} - h_{o,5}\right) = \dot{m}_c \left(h_{o,3} - h_{o,2}\right) = \dot{W}_c \tag{11.23}$$

If we assume $\dot{m}_t C_{P_t} = \dot{m}_c C_{P_c}$ then...

$$T_{o4} - T_{o5} = T_{o3} - T_{o2}$$

which gives us

$$T_{o5} \cong T_{o4} - \left(T_{o3} - T_{o2}\right) \tag{11.24}$$

Turbine Efficiency:

$$\eta_t == \frac{h_{o4} - h_{o5}}{h_{o4} - h_{o5,s}} \tag{11.25}$$

$$\frac{P_{o5}}{P_{o4}} = \left[1 - \frac{1}{\eta_t}\left(1 - \frac{T_{o5}}{T_{o4}}\right)\right]^{\frac{\gamma}{\gamma-1}} \tag{11.26}$$

Nozzle Inlet ⑥

with no A/B, $T_{o,6} = T_{o,5}$ and $P_{o,6} = P_{o,5}$

with A/B, energy equation and similar to the burner analysis above.

Nozzle Exit ⑦

$$\frac{u_e^2}{2} = h_{o7} - h_7 = \eta_n \left(h_{o7} - h_{7,s}\right) \tag{11.27}$$

$$\eta_n = \frac{h_{o7} - h_7}{h_{o7} - h_{7,s}} \tag{11.28}$$

If we have steady, adiabatic flow $\rightarrow \quad h_{o,7} = h_{o,6}$

$$u_e = \sqrt{2\,\eta_n \frac{\gamma}{\gamma - 1} R T_{o6} \left[1 - \left(\frac{P_7}{P_{o6}}\right)^{\frac{\gamma-1}{\gamma}}\right]} \tag{11.29}$$

$$\frac{P_{o6}}{P_7} = \frac{P_{o6}}{P_{o5}} \cdot \frac{P_{o5}}{P_{o4}} \cdot \frac{P_{o4}}{P_{o3}} \cdot \frac{P_{o3}}{P_{o2}} \cdot \frac{P_{o2}}{P_a} \cdot \frac{P_a}{P_7} \tag{11.30}$$

Observations from Turbojet Analysis (also see Handout 3D)

- Assumes component efficiencies gives in table
- Mach # variation with altitude also as given
- Note γ changes through engine
- Assume product of

Inputs to the model:

TABLE 5.1 Turbojet calculation parameters

Component	Adiabatic efficiency	Average specific heat ratio
Diffuser	$\eta_d = 0.97$	1.40
Compressor	$\eta_c = 0.85$	1.37
Burner	$\eta_b = 1.00$	1.35
Turbine	$\eta_t = 0.90$	1.33
Nozzle	$\eta_n = 0.98$	1.36
Fuel heating value, 45,000 kJ/kg		

Flight altitude (cruise Mach no.)	Ambient pressure (kPa)	Ambient temperature (K)
Sea level (0)	101.30	288.2
40,000 ft (12,200 m) (0.85)	18.75	216.7
60,000 ft (18,300 m) (2.0)	7.170	216.7
80,000 ft (24,400 m) (3.0)	2.097	216.7

Results shown in Figs. 5.19 to 5.22

Look at how specific thrust, $TSFC$ affected by M#, T_{o4} and r_c.

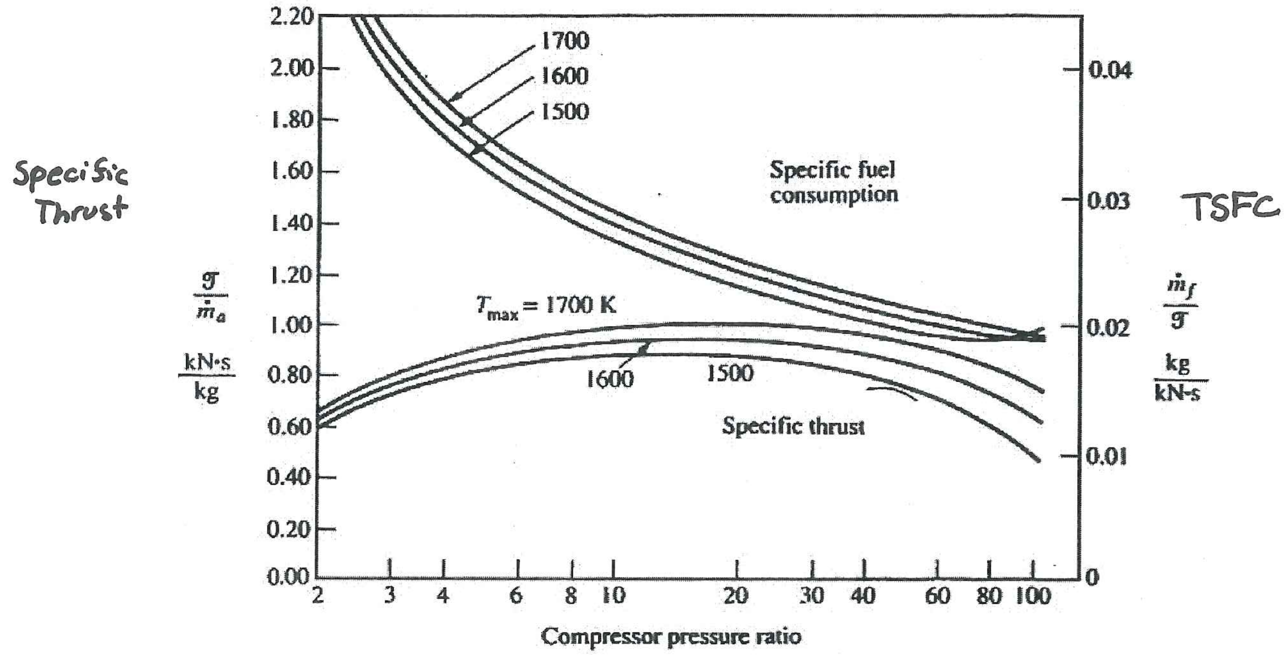


FIGURE 5.19 Turbojet static thrust and fuel consumption (M = 0).

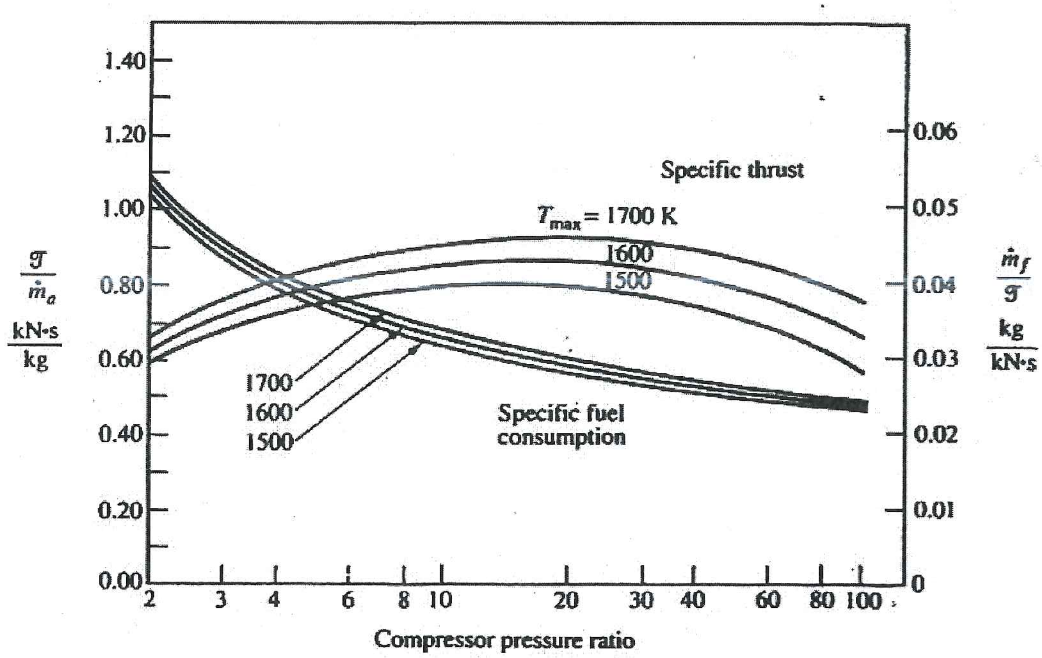


FIGURE 5.20 Turbojet cruise thrust and fuel consumption (M = 0.85).

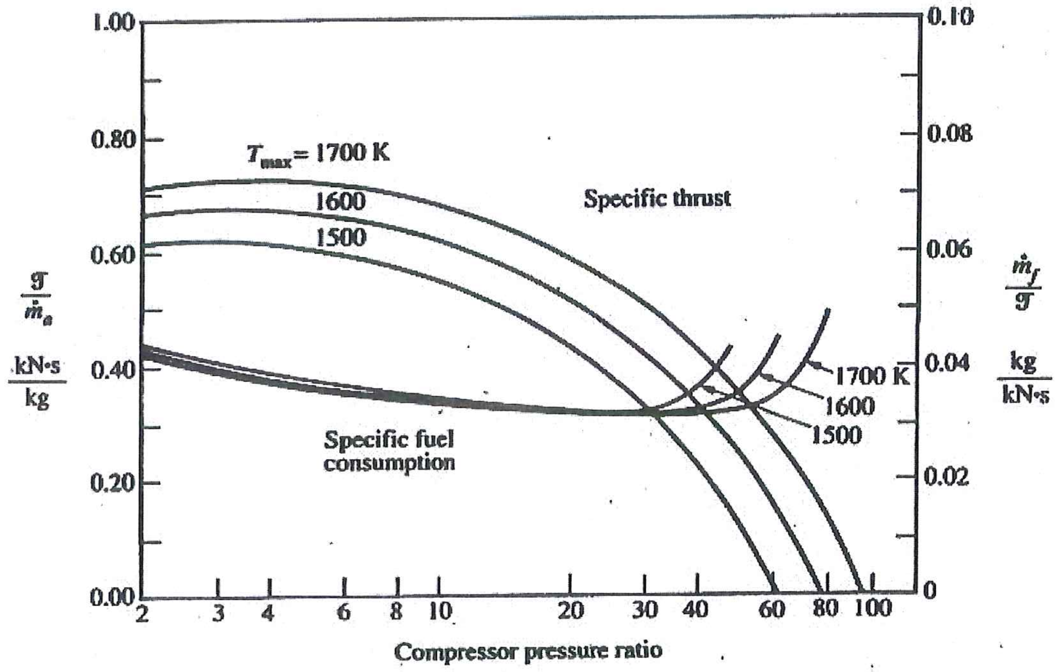


FIGURE 5.21 Turbojet cruise thrust and fuel consumption (M = 2).

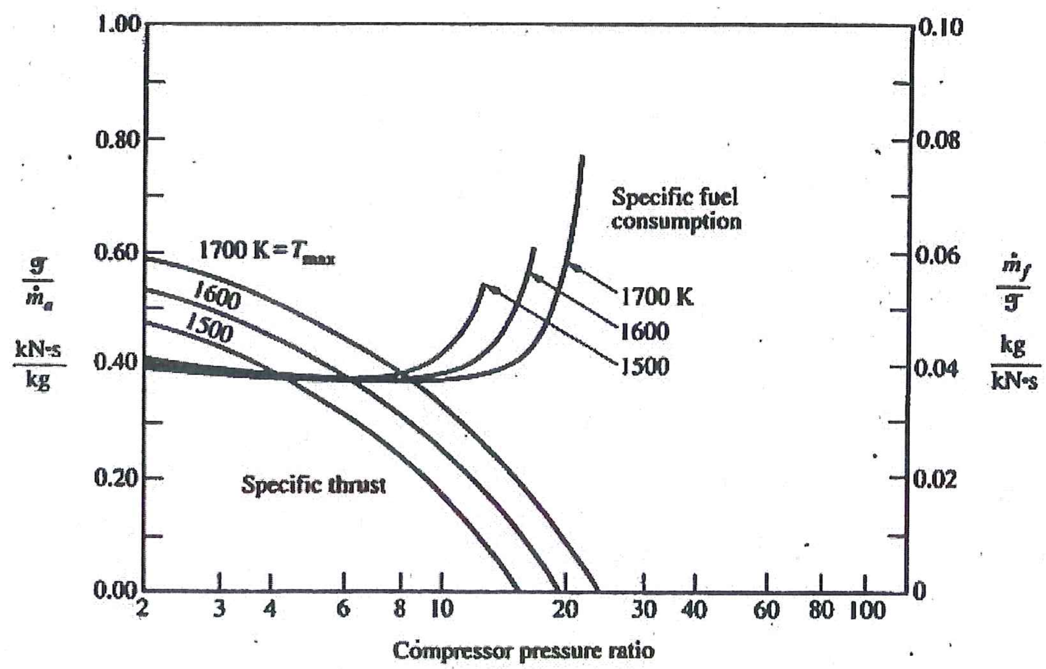


FIGURE 5.22 Turbojet cruise thrust and fuel consumption (M = 3).

Notice:

1. For a given M# and T_{o4}, the r_c that maximizes specific thrust does NOT minimize $TSFC$. Must compromise.
2. Raising T_{o4} can substantially improve specific thrust. $T_{\max} \sim 1700K$ due to material properties (this is below stoichiometric mixture ratio of hydrocarbon and air). Research focused on new high-temp metal alloys.
3. r_c required to minimize $TSFC$ much less for supersonic than subsonic flight. At $M = 3$, $r_c = 1$ for max specific thrust! Turbojet has become a ramjet.

In-Class Example Turbojet Problem

A turbojet engine has the following attributes:

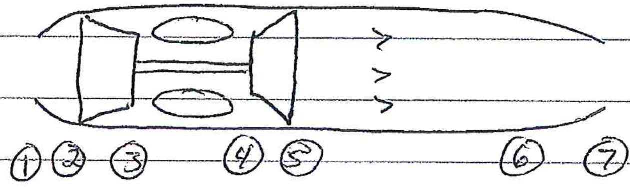
$$\eta_d = 0.90 \qquad \eta_c = 0.85 \qquad \eta_b = 0.99$$

$$\pi_b = 0.95 \qquad \eta_t = 0.90 \qquad \eta_n = 1$$

The heating value of the fuel (JP-8) is $43,400kJ/kg$. The turbojet is to be tested at static sea-level conditions ($T = 288K, P = 101.3kPa$) where its air mass flow rate is $270kg/s$. The nozzle is a converging type, the burner exit gas temperature is 1320 K, and the afterburners have been removed. Assume $C_p = 1,000J/kg - K$ and $\gamma = 1.4$ for the entire engine. If the compressor pressure ratio is 5-to-1, calculate:

1. The exhaust velocity and the optimum cruising speed of the turbojet based on the engine exhaust velocity [m/s]
2. The total engine thrust (jet and pressure thrusts) [N]
3. TSFC [kg/kN-s]
4. The exit nozzle area and diameter [m^2, m]
5. Specific thrust [kN-s/kg]

Turbojet Example



$T_{04} = 1320\,K$ $\quad \gamma = 1.4$ $\quad C_p = 1.0\ kJ/kg\text{-}K$ $\quad \frac{P_{03}}{P_{02}} = 5$

$\dot{m}_a = 270\ kg/s$ $\quad M_a = 0$ static

① + ② $P_1 = P_{01} = P_2 = P_{02} = 101.3\,kPa$

$T_1 = T_{01} = T_{02} = T_2 = 288\,K$

③ $P_{03} = 5 \cdot P_{02} = 506.5\,kPa$

$$T_{03} = T_{02}\left[1 + \frac{1}{\eta_c}\left\{r_c^{\frac{\gamma-1}{\gamma}} - 1\right\}\right] = 288\left[1 + \frac{1}{.85}\left\{5^{\frac{\gamma-1}{\gamma}} - 1\right\}\right]$$

~~= 404.9 K~~

$= 486\,K$

$$f = \frac{T_{04}/T_{03} - 1}{\frac{\eta_b Q_R}{C_p T_{03}} - T_{04}/T_{03}} = 0.0200$$

~~0.0198~~

④ $T_{04} = 1320\,K$ $\quad P_{04} = .95\,P_{03} = 481.2\,kPa$

⑤ $T_{05} \approx T_{04} - (T_{03} - T_{02}) = 1122\,K$

$$P_{05} = P_{04}\left[1 - \frac{1}{\eta_t}\left(1 - \frac{T_{05}}{T_{04}}\right)\right]^{\gamma/\gamma-1} = 254.2\,kPa$$

(6) $T_{06} = T_{05} = 1122K$

$P_{06} = P_{05} = 254.2\ kPa$

(7) ~~$u_e = \sqrt{2\eta_n \frac{\gamma}{\gamma-1} RT_{06}\left[1-\left(\frac{P_a}{P_{06}}\right)\right]}$~~

Nozzle Choked? $P_{crit} = P_{06}\left(\frac{2}{\gamma+1}\right)^{\gamma/\gamma-1} = 134.3\ kPa$

$P_{crit} > P_a$? yes, choked flow so $P_7 = P_{crit}$ $M_7 = 1$.

$$u_e = \sqrt{2\eta_n \frac{\gamma}{\gamma-1} RT_{06}\left[1-\left(\frac{P_7}{P_{06}}\right)^{\frac{\gamma-1}{\gamma}}\right]} = 612.9\ m/s$$

a) $u_e = 612.9\ m/s$ $u_{optimum} = \frac{u_e}{2} =$ ~~[illegible]~~ $= 306.5\ m/s$

$M = 0.9$

b) $F = \dot{m}(u_e - u) + A_e(P_e - P_a)$

$= \dot{m}_a[(1+f)u_e - u] + A_e(P_e - P_a)$ $\dot{m}_T = \dot{m}_a(1+f) = 275.4\ kg/s$

How to get A_e?

~~$\dot{m}^* = P_e A^* \left\{\frac{\gamma}{RT^*}\right\}\left[\frac{2}{\gamma+1}\right]^{\gamma/\gamma-1}$~~

$$\frac{\dot{m}_T}{A^*} = \frac{P_{06}}{\sqrt{RT_{06}}}\sqrt{\gamma}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow A^* = A_e = .898\ m^2$$

$= 306.7$

$D_e = 1.07\ m$

then

$$F = 270[(1+.02)\,612.9 - 0] + .898(134300 - 101300)$$
$$= 168793 \quad + \quad 29634$$
jet pressure
$$= 198427\ N \approx 200\ kN$$

c) TSFC $\frac{\dot{m}_f}{F} = \frac{(.02)(270)}{198427} = 2.72\times10^{-5}\ kg/N\text{-}s$

$$= .0272\ kg/kN\text{-}s$$

d) $A_e = .898\ m^2$ $D_e =$ ~~[illegible]~~ $1.07m$

e) $F/\dot{m}_a = \frac{198.427\ kN}{270\ kg/s} = .735\ \frac{kN\text{-}s}{kg}$

1.4 Turbofan

Schematic of a Turbofan with numbering convention

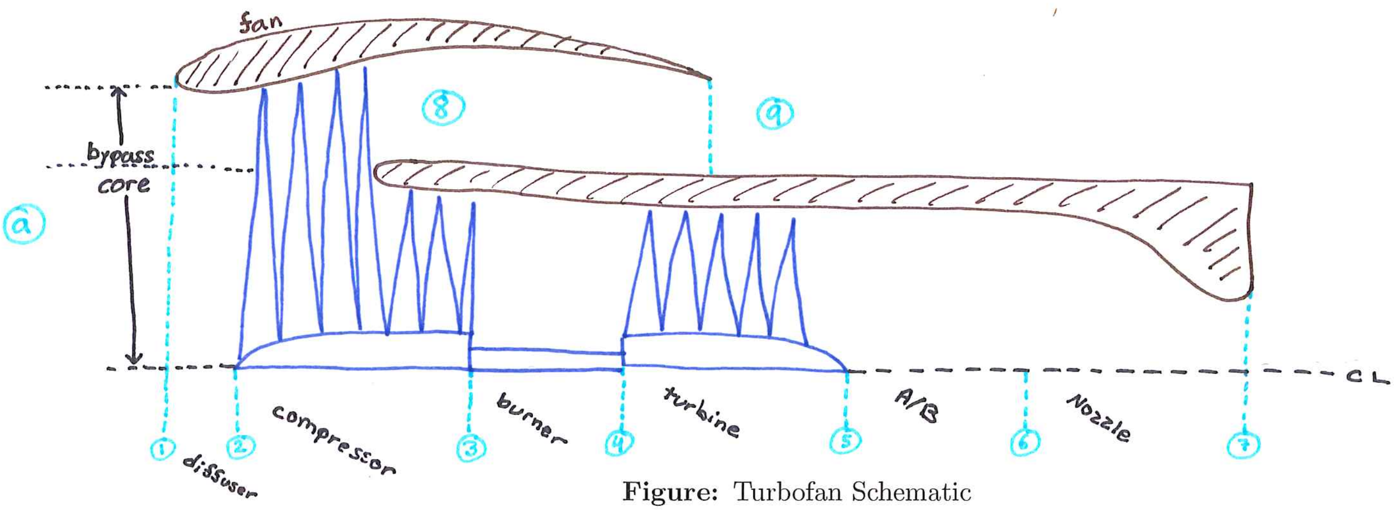


Figure: Turbofan Schematic

Bypass ratio

$$\beta = \frac{\dot{m}_{\text{bypass}}}{\dot{m}_{\text{core}}}$$

$$\dot{m}_{\text{total}} = \dot{m}_{\text{core}} + \dot{m}_{\text{bypass}} + \dot{m}_{\text{fuel}}$$

Core flow analysis identical to Turbojet for stages (1) to (4). We assume both core and bypass expand to ambient pressure, separately. Let's now look at how a bypass modifies the analysis.

New Thrust Eqn:

$$F = (\dot{m}_{\text{core}} + \dot{m}_{\text{f}})\, u_e + \dot{m}_{\text{bypass}}\, u_{ef} - (\dot{m}_{\text{core}} + \dot{m}_{\text{bypass}})\, u$$

Specific Thrust

$$\frac{F}{\dot{m}_{\text{core}}} = \frac{F}{\dot{m}_{\text{a}}} = (1+f)\, u_e + \beta\, u_{ef} - (1+\beta)\, u \tag{11.31}$$

Where

$$u_e = \text{Core exit velocity}$$

$$u_{ef} = \text{Fan exit velocity}\dot{m}_{\text{core}}$$

TSFC

$$\frac{\dot{m}_\mathrm{f}}{F} = \frac{f}{\frac{F}{\dot{m}_\mathrm{a}}} = \frac{f}{\frac{F}{\dot{m}_\mathrm{core}}} \tag{11.32}$$

Now, step through the engine calculating the thermodynamic state variables at each location, to eventually find the exit conditions.

Core stages (1) to (4) same as Turbojet

Fan Inlet

Fan and core receive air from same diffuser, so conditions at fan entrance are state (4) 2, from Turbojet:

$$\frac{T_{o2}}{T_a} = \left(1 + \frac{\gamma - 1}{2} M_a^2\right)$$

$$P_{o2} = P_a \left[1 + \eta_d \left(\frac{T_{o2}}{T_a} - 1\right)\right]^{\frac{\gamma}{\gamma-1}}$$

Fan Exit

Fan pressure ratio (input variable): $r_f = \frac{P_{o8}}{P_{o2}}$

$$\eta_f = \text{Fan Adiabatic Efficiency}$$

$$\eta_f = \frac{h_{o8,s} - h_{o2}}{h_{o8} - h_{o2}} \tag{11.33}$$

$$T_{o8} = T_{o2} \left[1 + \frac{1}{\eta_f} \left(r_f^{\frac{\gamma-1}{\gamma}} - 1\right)\right] \tag{11.34}$$

Fan Nozzle Exit

$$u_{ef} = \sqrt{2\frac{\gamma}{\gamma - 1}\eta_{fn} R T_{o8}\left[1 - \left(\frac{P_a}{P_{o8}}\right)^{\frac{\gamma-1}{\gamma}}\right]} \quad (11.35)$$

Where

$$\eta_{fn} = \text{Fan nozzle efficiency}$$

We use P_a because this is an idea case, otherwise we check to see if there is choked flow and then use P_e.

Turbine Outlet

Now turbine has to drive compressor AND fan.

Cons. Of Energy:

$$\dot{m}_{\text{t}}\, C_{p,t}(T_{o4} - T_{o5}) = \dot{m}_{\text{core}}\, C_{p,c}(T_{o3} - T_{o2}) = \dot{m}_{\text{bypass}}\, C_{p,f}(T_{o8} - T_{o2}) \quad (11.36)$$

↖ Turbine ↖ compressor ↖ fan

Assume:

$$\dot{m}_{\text{t}}\, C_{p,t} = \dot{m}_{\text{core}}\, C_{p,c} = \dot{m}_{\text{bypass}}\, C_{p,f} \quad (11.37)$$

$$T_{o5} \cong T_{o4} - (T_{o3} - T_{o2}) - \beta\,(T_{o8} - T_{o2}) \quad (11.38)$$

$$\frac{P_{o5}}{P_{o4}} = \left[1 - \frac{1}{\eta_t}\left(1 - \frac{T_{o5}}{T_{o4}}\right)\right]^{\frac{\gamma}{\gamma-1}}$$

Core Nozzle Inlet

No A/B afterburner → $T_{o6} = T_{o5}$ and $P_{o6} = P_{o5}$

With A/B afterburn → do another combustor/burner analysis

Core Nozzle Exit

Same as before, just like Eqn. (11.29)

$$u_e = \sqrt{2 \frac{\gamma}{\gamma - 1} \eta_n R T_{o6} \left[1 - \left(\frac{P_a}{P_{o6}}\right)^{\frac{\gamma-1}{\gamma}}\right]}$$

Again, $P_e = P_7 = P_a$ if an ideal case. If NOT ideal, then we do a choked flow nozzle analysis.

Observations from Turbofan Analysis (also see Handout 3d)

-same input data as section XI. B. Turbojet for the CORE

TABLE 5.1 Turbojet calculation parameters

Component	Adiabatic efficiency	Average specific heat ratio
Diffuser	$\eta_d = 0.97$	1.40
Compressor	$\eta_c = 0.85$	1.37
Burner	$\eta_b = 1.00$	1.35
Turbine	$\eta_t = 0.90$	1.33
Nozzle	$\eta_n = 0.98$	1.36
Fuel heating value, 45,000 kJ/kg		

Flight altitude (cruise Mach no.)	Ambient pressure (kPa)	Ambient temperature (K)
Sea level (0)	101.30	288.2
40,000 ft (12,200 m) (0.85)	18.75	216.7
60,000 ft (18,300 m) (2.0)	7.170	216.7
80,000 ft (24,400 m) (3.0)	2.097	216.7

-Assumed TurboFan properties are:

Component	Efficiency	Gamma
Diffuser	0.97	1.4
Fan	0.85	1.4
Fan Nozzle	0.97	1.4
Fan Pressure Ratio	$r_f = 1.5$	

Consider $M = 0$ case:

Turbojet on top ($\beta = 0$), Turbofan on bottom ($\beta = 5$)

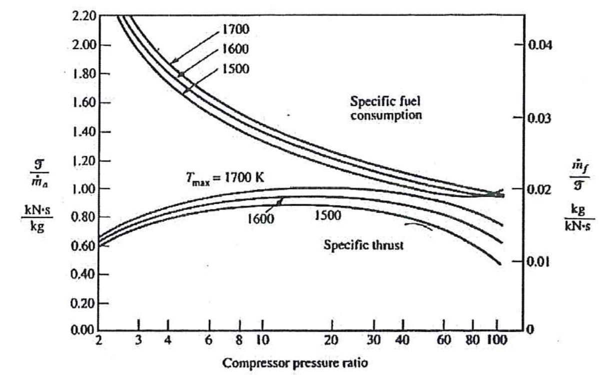


FIGURE 5.19 Turbojet static thrust and fuel consumption (M = 0).

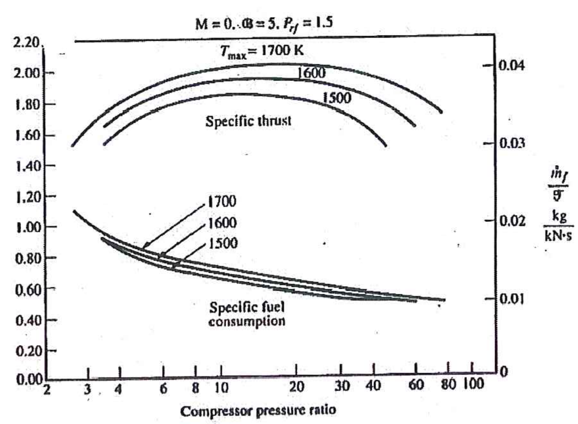


FIGURE 5.29 Turbofan static thrust and fuel consumption.

Note: specific thrust nearly doubles for Turbofan (goes from 1 for turbojet to 2 for turbofan), and TSFC decreases significantly (nearly half, 0.025 for turbojet, drops to 0.015 for turbofan)

Compare $M = 0.85$

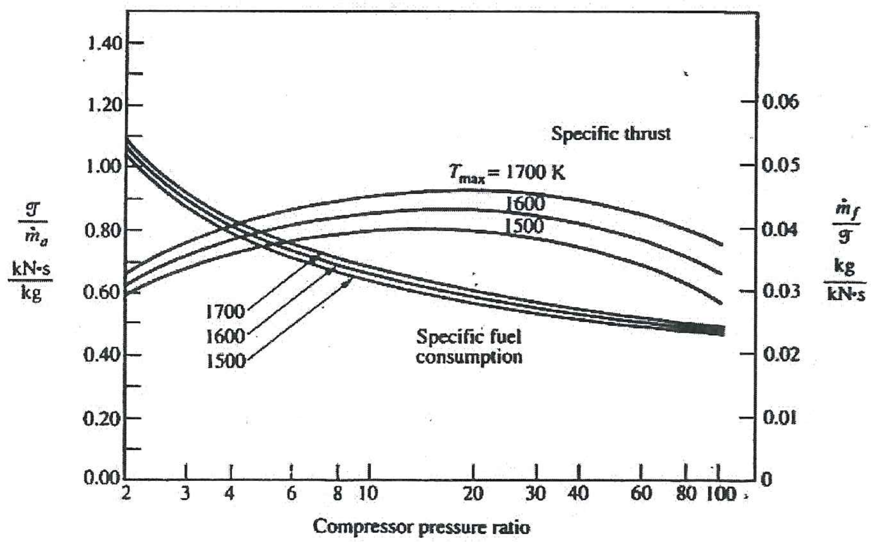


FIGURE 5.20 Turbojet cruise thrust and fuel consumption (M = 0.85).

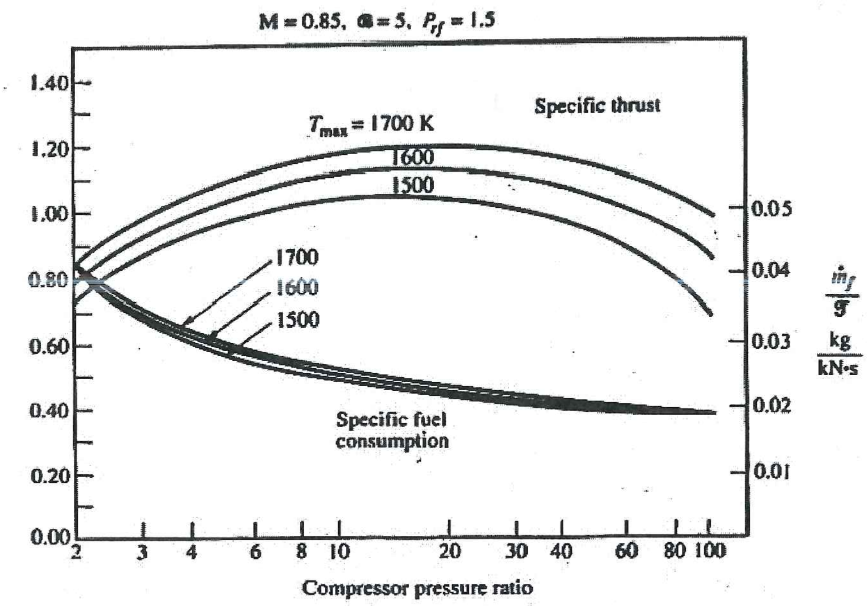


FIGURE 5.30 Turbofan cruise thrust and fuel consumption.

Note: Same trends as the $M = 0$ case above, but improvements not as substantial. Specific thrust increases from 0.9 to 1.2, TSFC decreases from 0.03 to 0.02.

Comparison of Efficiencies at $M = 0.85$:

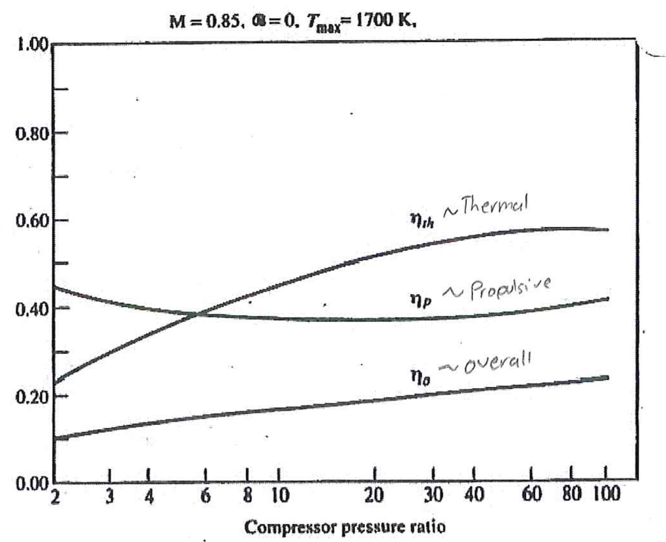


FIGURE 5.31 Turbojet thermal and propulsion efficiencies.

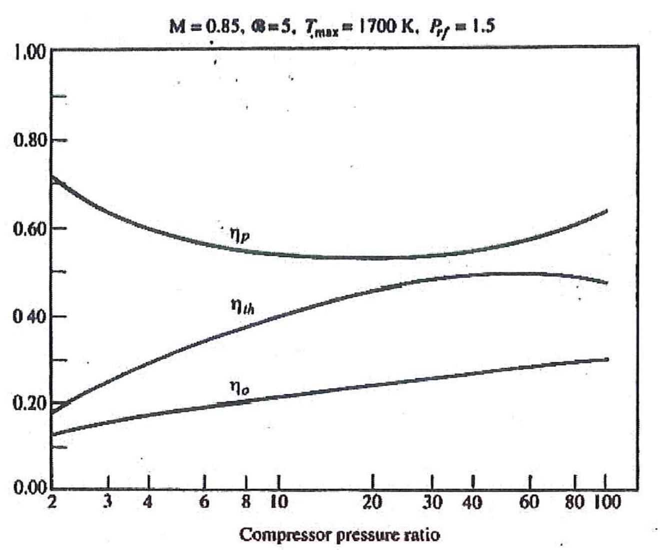


FIGURE 5.32 Turbofan thermal and propulsion efficiencies.

Turbofan efficiency greater than Turbojet, 20-30% for turbofan vs. 15-20% for turbojet

EXAMPLE Problem:

Consider an ideal turbofan engine with the conditions $M_a = 2$, $r_c = 20$, $r_f = 2.2$, $\beta = 3$, $\gamma = 1.4$, and $T_{o4}/T_{oa} = 7$. Find the Mach number at the exit of the core and fan. (Notice: No ambient temperature or pressure given!, not necessary to find M#, but it is necessary to find exhaust velocity and performance)

1) $M_a = 2 \quad r_c = 20 \quad r_f = 2.2 \quad \beta = 3 \quad \gamma = 1.4$

$$\frac{T_{o4}}{T_{oa}} = 7 = \frac{T_{o4}}{T_{o2}}$$

(o2) $$\frac{T_{oa}}{T_a} = \left(1 + \frac{\gamma-1}{2} M_a^2\right) = \frac{T_{o2}}{T_a} \qquad \frac{P_{o2}}{P_a} = \frac{P_{oa}}{P_a} = \left(1 + \frac{\gamma-1}{2} M_a^2\right)^{\frac{\gamma}{\gamma-1}} = 7.82$$

(o3) $$\frac{P_{o3}}{P_{o2}} = 20 \qquad \frac{T_{o3}}{T_{o2}} = 20^{\frac{\gamma-1}{\gamma}} = 2.35$$

(o4) $$P_{o4} = P_{o3} \qquad \frac{T_{o4}}{T_{o3}} = \underbrace{\frac{T_{o4}}{T_{oa}}}_{7} \cdot \underbrace{\frac{T_{oa}}{T_{o2}}}_{1} \cdot \underbrace{\frac{T_{o2}}{T_{o3}}}_{} = 2.98$$

$$\frac{P_{o4}}{P_{o3}} = 1$$

(o8) $$\frac{P_{o8}}{P_{o2}} = 2.2 \qquad \frac{T_{o8}}{T_{o2}} = 2.2^{\frac{\gamma-1}{\gamma}} = 1.25$$

(o5) $$\frac{T_{o5}}{T_{o4}} = 1 - \frac{T_{o3}}{T_{o4}} + \frac{T_{o2}}{T_{o4}} - \beta\left(\frac{T_{o8}}{T_{o4}} - \frac{T_{o2}}{T_{o4}}\right)$$

$$= 1 - \frac{1}{2.98} + \frac{1}{7} - 3\left(\underbrace{\frac{T_{o8}}{T_{o2}}}_{1.25} \cdot \underbrace{\frac{T_{o2}}{T_{o4}}}_{\frac{1}{7}} - \frac{1}{7}\right) = 0.70$$

$$\frac{P_{o5}}{P_{o4}} = \left(\frac{T_{o5}}{T_{o4}}\right)^{\frac{\gamma}{\gamma-1}} = 0.287$$

M_e core $$\frac{P_{05}}{P_e} = \left(1+\frac{\gamma-1}{2}M_e^2\right)^{\frac{\gamma}{\gamma-1}} = \frac{P_{05}}{P_a}$$

$$\frac{P_{05}}{P_a} = \frac{P_{05}}{P_{04}} \cdot \frac{P_{04}}{P_{03}} \cdot \frac{P_{03}}{P_{02}} \cdot \frac{P_{02}}{P_a} = (.287)(1)(20)(7.82) = 44.89$$

so $M_{e,core} = 3.13$

$M_{e,\,fan}$ $$\frac{P_{08}}{P_e} = \frac{P_{08}}{P_a} = \left(1+\frac{\gamma-1}{2}M_{e,f}^2\right)^{\frac{\gamma}{\gamma-1}}$$

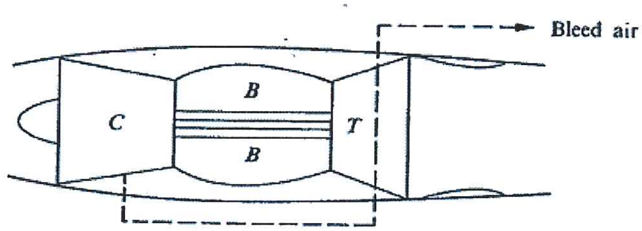
$$\frac{P_{08}}{P_a} = \frac{P_{08}}{P_{02}} \cdot \frac{P_{02}}{P_a} = (2.2)(7.82) = 17.2$$

so $M_{e,\,fan} = 2.50$

ADDITIONAL EXAMPLE PROBLEM

Consider two versions of a jet engine. The first is a standard engine run with a turbine inlet temperature of 1200 K. The second is identical, except that its turbine is cooled by bleeding air from the compressor; in this way the allowable turbine inlet temperature may be raised to 1600 K. The basic engine and the bleed air line are indicated in the figure. To make an estimate of the effect of this modification on the thrust, suppose that the engine is flying at Mach 2 (there's a normal shock present) at an altitude where the ambient temperature is 200 K and that compressor pressure ratio is 9:1. Suppose also that for the second engine 10% of the airflow is bled from the compressor at a point where the pressure ratio is 9:1 (i.e., after its been fully compressed). After cooling the turbine, the bleed air is exhausted from the engine with no appreciable velocity. For simplicity, assume all components of both engines to be reversible. Let $\gamma = 1.4$ and $c_p = 1.0$ kJ/kg-K.

a. Determine the thrust per unit mass flow of air entering the compressor (specific thrust) for each engine.
b. What is the ratio of the thrust specific fuel consumption (TSFC) of the second engine to that of the first?



$r_c = 9 \qquad M_a = 2 \qquad T_a = 200\ K \qquad \gamma = 1.4$

$u = M\sqrt{\gamma R T_a} = 567\ m/s \qquad c_p = 1004\ J/kg\text{-}K$

1st ② @ $M_a = 2$ $\quad \frac{T_{oa}}{T_a} = 1.80 \Rightarrow T_{oa} = T_{o2} = 360K$

$$\frac{P_{oa}}{P_a} = \left(\frac{T_{oa}}{T_a}\right)^{\frac{\gamma}{\gamma-1}} = 7.82$$

normal shock @ M = 2 $\quad P_{o2}/P_{oa} = .721$

③ $\frac{T_{o3}}{T_{o2}} = \left(\frac{P_{o3}}{P_{o2}}\right)^{\frac{\gamma-1}{\gamma}} = 9^{\frac{.4}{1.4}} \Rightarrow T_{o3} = 674\ K$

④ $T_{o4} = 1200\ K$

⑤ $T_{o5} = T_{o4} - (T_{o3} - T_{o2}) = 886\ K$

ⓔ $u_e = \sqrt{2c_p(T_{o5} - T_e)} = \sqrt{2c_pT_{o5}\left(1 - \left(\frac{P_e}{P_{o5}}\right)^{\frac{\gamma-1}{\gamma}}\right)}$

assume $P_e = P_a$ matched $\Rightarrow$

$$\frac{P_e}{P_{o5}} = \frac{P_a}{P_{o5}} = \frac{P_a}{P_{oa}} \cdot \frac{P_{oa}}{P_{o2}} \frac{P_{o2}}{P_{o3}} \frac{P_{o3}}{P_{o4}} \frac{P_{o4}}{P_{o5}}$$

$$= \frac{1}{7.82} \frac{1}{.721} \cdot \frac{1}{9} \cdot 1 \cdot 1 = 0.0197$$

$u_e = 1095\ m/s$

2^{nd} w/ bleed

(2) $T_{02} = 360\ K \qquad \frac{P_{0a}}{P_a} = 7.82 \qquad \frac{P_{02}}{P_{0a}} = .721$

(3) $T_{03} = 674\ K \qquad P_{03}/P_{02} = 9$

(4) $T_{04} = 1600\ K$

(5) $\dot{m}_c C_p (T_{03} - T_{02}) = \dot{m}_t C_p (T_{04} - T_{05})$

$$T_{05} = T_{04} - \frac{\dot{m}_c}{\dot{m}_t}(T_{03} - T_{02}) \qquad \dot{m}_t = 0.9\,\dot{m}_c$$

$$= T_{04} - \frac{1}{0.9}(T_{03} - T_{02})$$

$$= 1251\ K$$

(e) $\frac{P_e}{P_{05}} = \frac{P_a}{P_{05}} = 0.0197 \qquad$ assuming $P_e = P_a$

$u_e = 1302\ m/s$

---

a) $\left.\frac{F}{\dot{m}_a}\right)_{case\,1} = \frac{\dot{m}_e u_e - \dot{m}_a u}{\dot{m}_a} \qquad$ w/ $\dot{m}_e = \dot{m}_a$

$$= u_e - u = 528\ \frac{N}{kg/s}$$

$\left.\frac{F}{\dot{m}_a}\right)_{case\,2} = 0.9\,u_e - u = 605\ \frac{N}{kg/s} \qquad$ now $\dot{m}_e = 0.9\,\dot{m}_a$

b) $TSFC = \frac{f}{F/\dot{m}_a}$ ← from (a) what about f ?

case 1 $\dot{m}_a C_p T_{03} + \dot{m}_f Q_R = (\dot{m}_a + \dot{m}_f) C_p T_{04}$

case 2 $.9\dot{m}_a C_p T_{03} + \dot{m}_f Q_R = (0.9\dot{m}_a + \dot{m}_f) C_p T_{04}$

case 1 $f Q_R = (1+f) C_p T_{04} - C_p T_{03}$ assume $f \ll 1$

$f Q_R = C_p (T_{04} - T_{03})$

case 2 $f Q_R = (0.9 + f) C_p T_{04} - 0.9 C_p T_{03}$ assume $f \ll 0.9$

$f Q_R = 0.9 C_p (T_{04} - T_{03})$

Now $$\frac{TSFC)_2}{TSFC)_1} = \frac{f_2}{f_1} \frac{F/\dot{m}_a)_1}{F/\dot{m}_a)_2} = \frac{.9 (T_{04} - T_{03})_2}{(T_{04} - T_{03})_1} \cdot \frac{528}{605}$$

$$= \frac{.9 (1600 - 674)}{(1200 - 674)} \cdot \frac{528}{605} = \boxed{1.38}$$

Section 3: Air Breathing Propulsion

AE435

Spring 2018

1 Component Performance

Contents

1.1 Inlets

Subsonic vs. Supersonic Inlets A Good Inlet:

1. Reduces Mach #
2. Provides uniform flow
3. Minimizes loss of stagnation pressure (Shocks, seperation and friction)

1.1.1 Subsonic Inlets (a.k.a. diffuser)

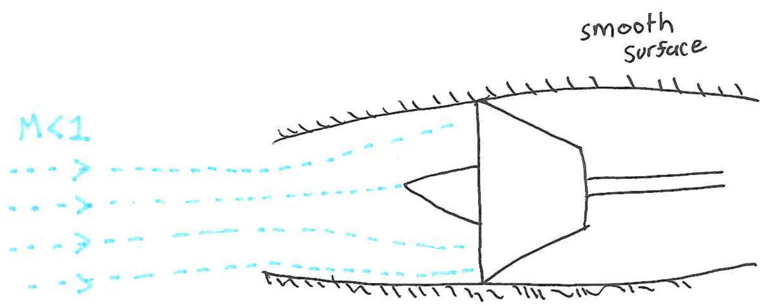


Figure: Subsonic Inlet

- Design nozzle inner and outer surfaces to minimize adverse pressure gradient (leads to separation)
- Reduce separation which leads to
  - Decreased total pressure, P_o, flow no longer uniform inside engine
  - Drag increase

Consider a subsonic inlet with the following stations:

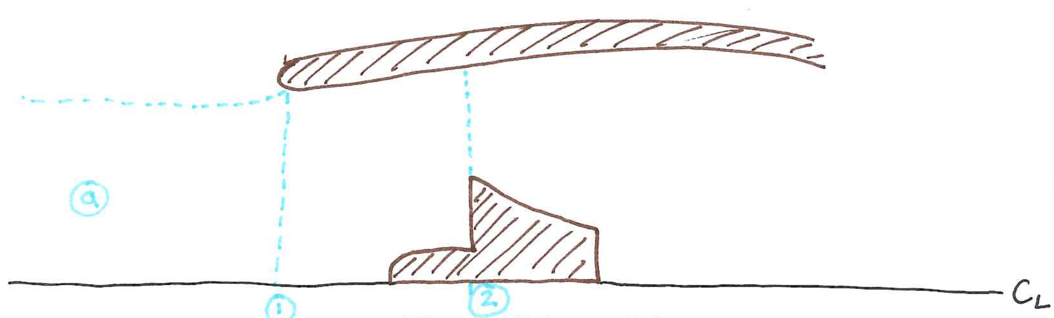


Figure: Subsonic Inlet

ⓐ - Ambient, ① - Entrance to inlet , ② - Entrance to compressor

(a) to (1) external deceleration of the flow (isentropic)

(1) to (2) internal deceleration, static P increases, total P_o drops (not isentropic)

If (a) to (2) WAS isentropic, we would arrive at state (2s)

Efficiency of diffuser inlet is then:

$$\eta_d = \frac{\text{Work Required Isentropically}}{\text{Actual Work Required}} = \frac{h_{2,s} - h_a}{h_2 - h_a} \tag{12.1}$$

No heat added, so T_o = constant, $T_{oa} = T_{o2}$, so:

$$\frac{T_2}{T_a} = \frac{1 + \frac{\gamma-1}{2} M_a^2}{1 + \frac{\gamma-1}{2} M_2^2} \qquad M_2^2 \ll 1 \text{Flow Decelerated}$$

$$T_2 \cong T_a \left(1 + \frac{\gamma - 1}{2} M_a^2\right) = T_{oa} = T_{o2} \tag{12.2}$$

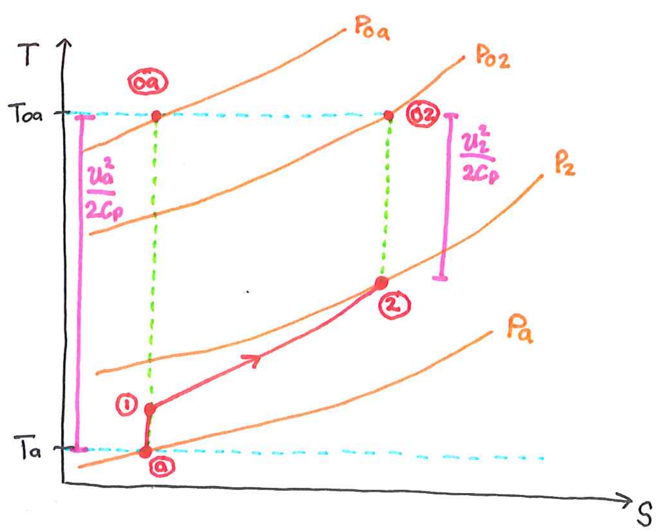
Also, (12.2) applies for the isentropic state (2s) since T_o = constant for isentropic process, so

$$T_{o2s} = T_{2s} \tag{12.3}$$

(o2s) is state reached by isentropic compression to actual P_{o2}. Assuming C_p = constant, (12.1) becomes

$$\eta_d = \frac{T_{2s} - T_a}{T_2 - T_a} = \frac{T_{o2s} - T_a}{T_{oa} - T_a} = \frac{\frac{T_{o2s}}{Ta} - 1}{\frac{T_{oa}}{T_a} - 1} \tag{12.4}$$

$$\frac{T_{oa}}{T_a} = 1 + \frac{\gamma - 1}{2} M^2 \qquad \frac{T_{o2s}}{T_a} = \left(\frac{P_{o2}}{P_a}\right)^{\frac{\gamma-1}{\gamma}} \tag{12.5}$$



Then (12.4) becomes

$$\eta_d = \frac{\left(\frac{P_{o2}}{P_a}\right)^{\frac{\gamma-1}{\gamma}} - 1}{\frac{\gamma-1}{2} M^2} \tag{12.6}$$

With diffuser pressure ratio defined as:

$$r_d = \frac{P_{o2}}{P_{oa}} \tag{12.7}$$

$$\eta_d = \frac{\left(1 + \frac{\gamma-1}{2} M^2\right) r_d^{\frac{\gamma-1}{\gamma}} - 1}{\frac{\gamma-1}{2} M^2} \tag{12.8}$$

Both are functions of Mach #.

Typical values for r_d and η_d from (12.8):

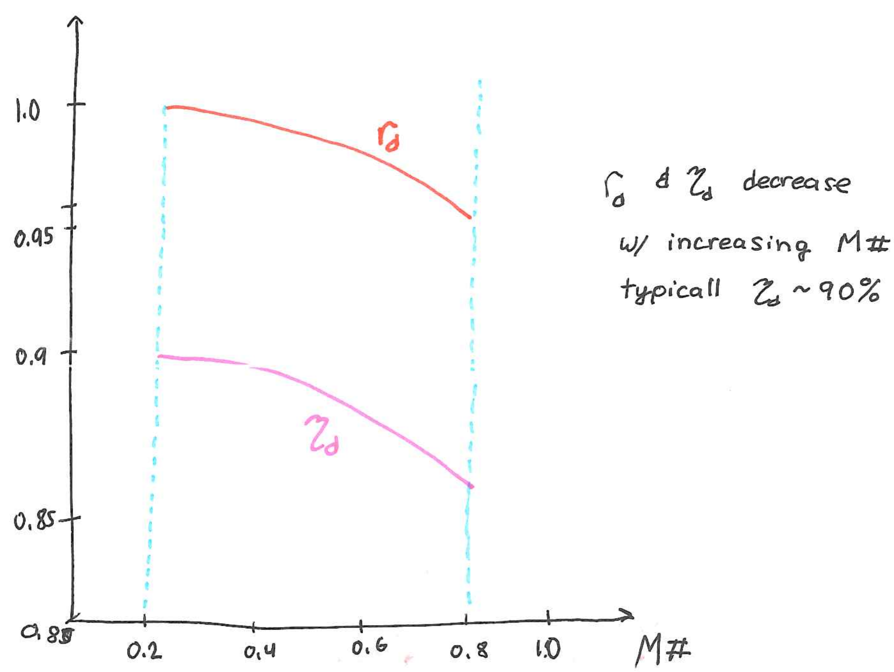


Figure: Mach V. r_d and η_d

Real vs. Ideal Compression-Expansion T-S Diagram

1. Gas compressed from 200-1200 K isentropically. To get the same pressure ratio in the presence of real effects, what is the real T2 likely to be? (900, 1100, or 1300 K?) For 90% efficiency?

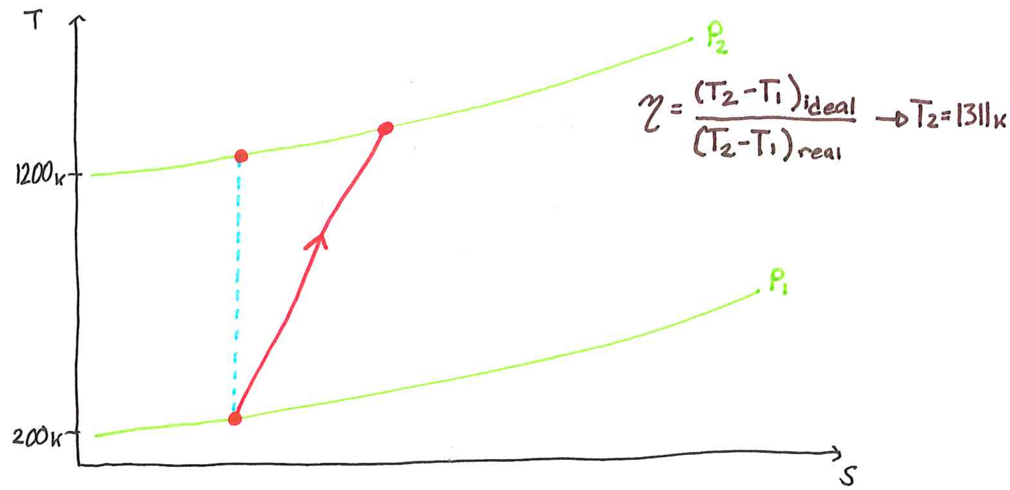


Figure: T-S Diagram

2. Gas expanded from 1200 to 200 K isentroipcally. To get the same pressure ratio in the presence of real effects, what is the real T2 likely to be ? (100, 300, 500 K?)

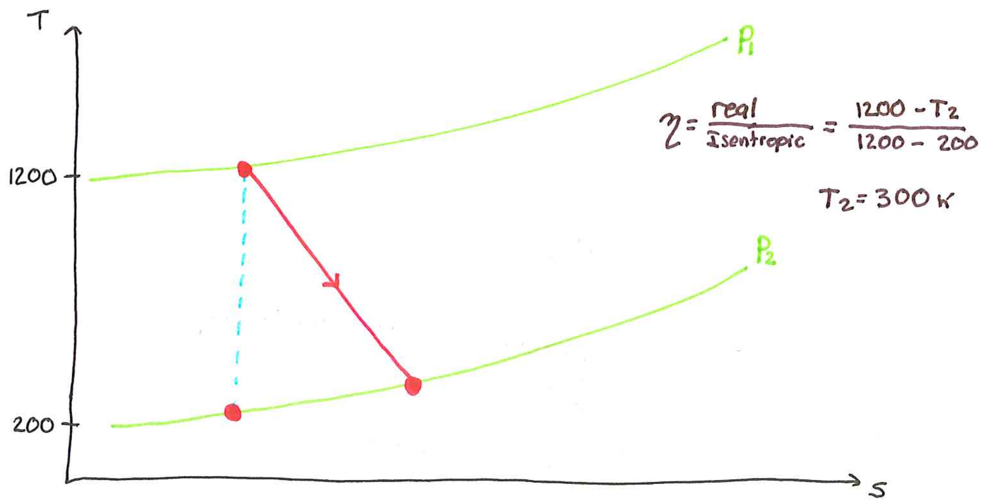


Figure: T-S Diagram

1.1.2 Supersonic Inlets

Let's start by considering a subsonic inlet flying at supersonic speeds. Get a normal shock wave out front.

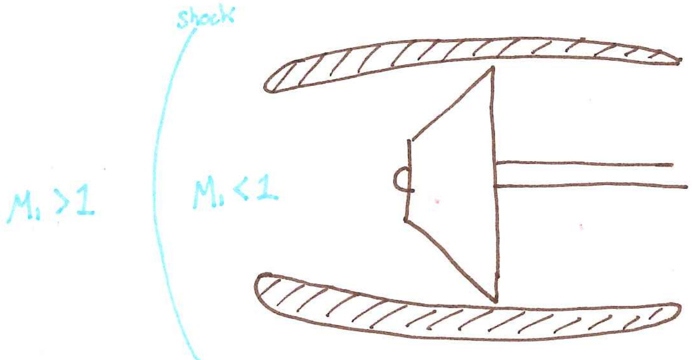


Figure: Supersonic Inlet

Applying normal shock relations (see section IV. B.), one would find:

Mach #	P_{o2}/P_{o1}
1.5	.93
2	0.70
2.5	0.5
3	0.31
4	0.15

For $M >\sim 1.5$, normal shock very bad, significant loss in total pressure (significant increase in entropy!). So for $M >\sim 1.5$, replace normal shock with oblique shock(s) followed by a normal shock.

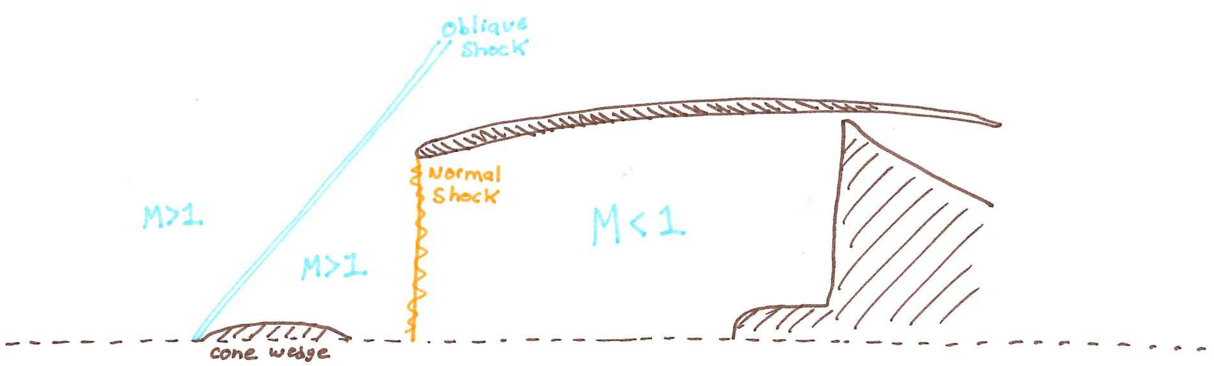


Figure: Supersonic Inlet with an Oblique Shock

- Oblique shocks have less total pressure loss than a normal shock
- In fact, a series of oblique shocks followed by a normal shock has less total pressure loss than a single normal shock
- Losses decrease as number of oblique shocks increases.

Example:

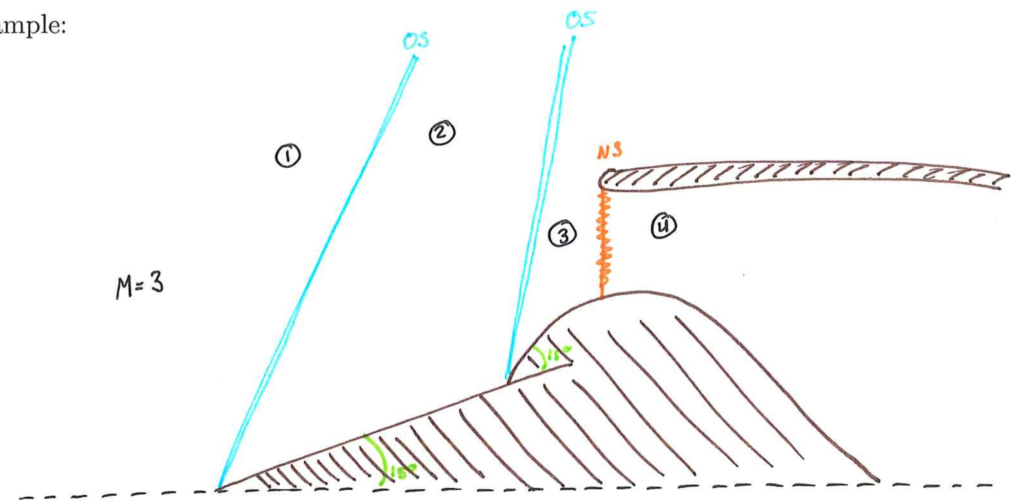


Figure: Supersonic Inlet with multiple Oblique Shocks

Two 15deg wedges, followed by 1 normal shock

State	Mach #	P_{o2}/P_{o1}
1	3	
2	2.26	0.895
3	1.65	0.945
4	0.67	0.870

Total pressure ratio across shock system: $P_{o4}/P_{o1} = 0.735$ (By multiplying all the pressure ratios.) Notice there is only a 25% pressure loss.

Compare with a single normal shock at $M = 3$: $P_{o4}/P_{o1} = 0.33$. This is a 70% pressure loss. This means we recover about 40% of pressure simply by introducing the oblique shocks.

Optimum wedge angles and effect of total number of shocks (see handout 3f):

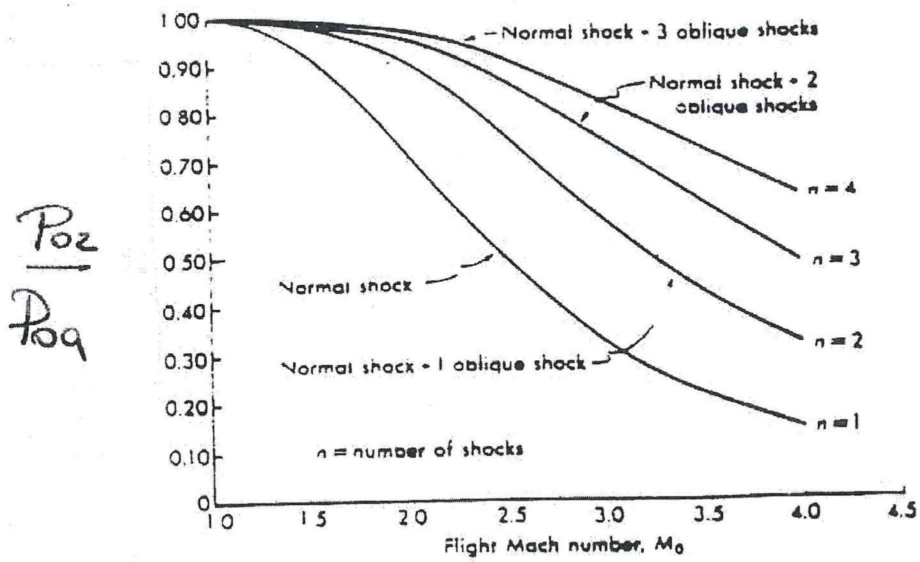
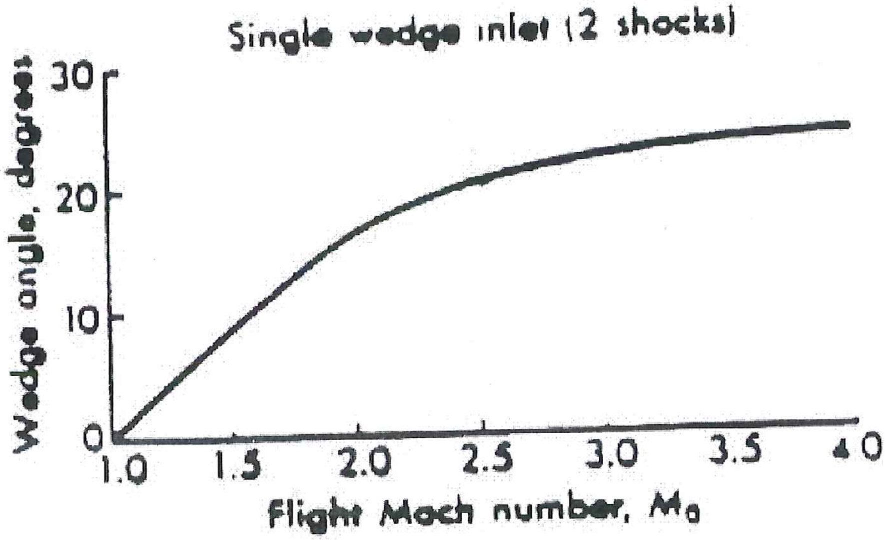


FIG. 5.12 Effect of number of shocks on total pressure recovery. (*From Ref. 8.*)

The above figure shows the pressure ratio across a shock system with optimum wedge angles for the given Flight Mach #. $n = 1$ is a single normal shock, $n = 2$ is one oblique followed by one normal shock, and so on.

The optimum wedge angles are given below. Optimum wedge angle provides the maximum total pressure ratio across the shock system.

First, for a single wedge inlet (one oblique shock followed by one normal shock)



Next, a double wedge inlet (two oblique shocks followed by one normal shock, similar to our example). Note the 2nd wedge angle is plotted as the difference from the first wedge angle, so mathematically:

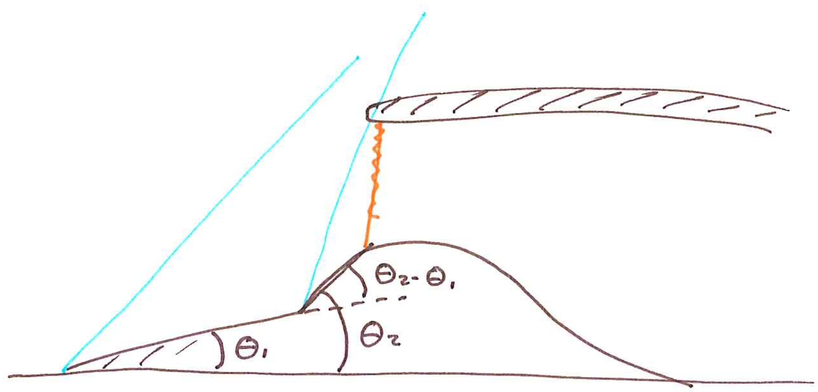
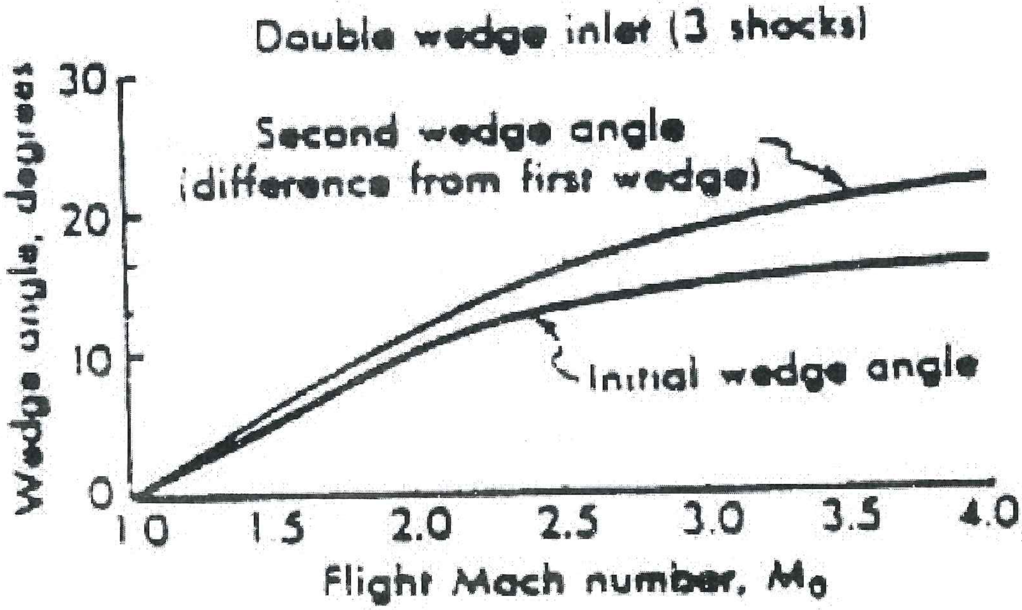
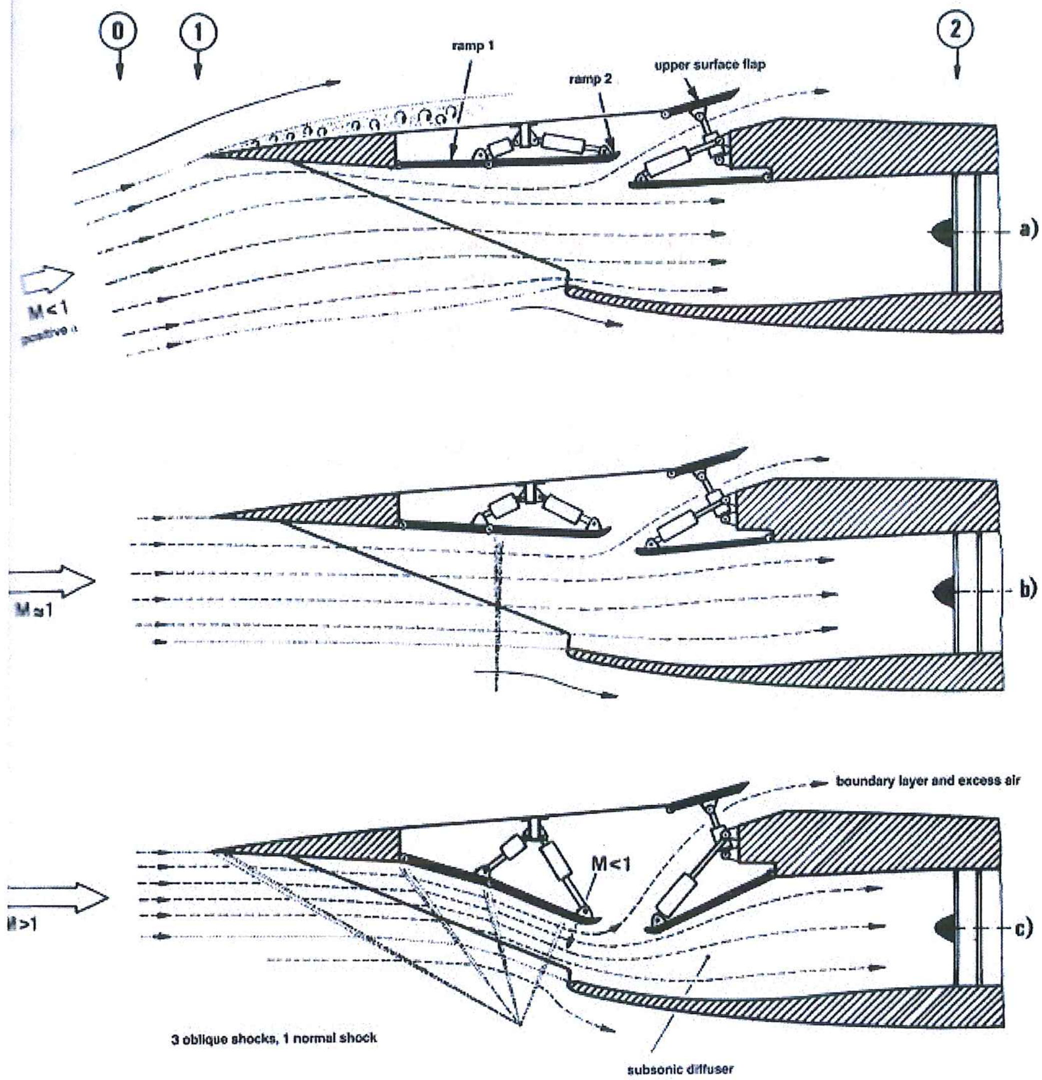


Figure: Wedge angles for supersonic inlets

Value in the $plot = \theta_2 - \theta_1$ A figure illustrating these angles in a double wedge inlet is given below.


Double wedge inlet (3 shocks)
Wedge angle, degrees
30
20
10
0
Second wedge angle
(difference from first wedge)
Initial wedge angle
1.0
1.5
2.0
2.5
3.0
3.5
4.0
Flight Mach number, M0



426 **Inlet of the F-14**
Ramp positions and flow at various flight conditions: a) subsonic speed and high angle-of-attack (typical manoeuvre case); b) transonic flow; c) supersonic flow.

1.2 Axial Compressors

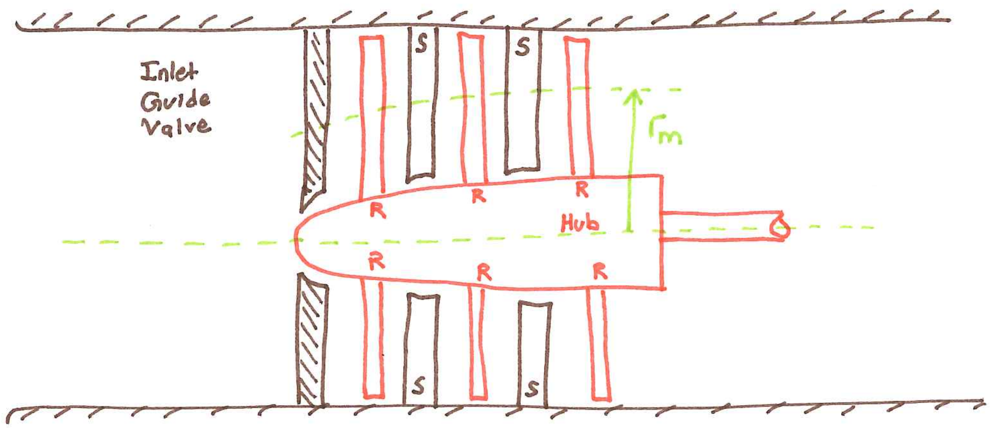


Figure: Compressor Diagram

- Rows of closely space rotor (R) blades are followed by rows of closely spaced stator (S) blades
- Rotor blades rotate with the hub, which is connected to the shaft, stator is stationary
- One rotor-stator pair makes a single compressor stage
- Rotor puts in azimuthal KE, stator removes it and increases static pressure
- Blade height is less at back of compressor to keep velocity at each stage approximately equal.
- As flow compressed, $\rho \uparrow$, to keep velocity the same, continuity requires $A \downarrow$. Think of it like $\rho_1 A_1 u_1 = \rho_2 u_2 A_2$ for $u_1 = u_2$, $\rho_1 < \rho_2$, $A_2 < A_1$
- Want same velocity so each stage blade size can be same, ease of design and manufacturing.
- r_m - mean radius, half the distance between the hub and housing (or rotor tip)

1.2.1 Work and Compression

Recognize that the velocity (magnitude and direction) of the air changes as it passes through one compressor stage. The compressor does work on the air, which causes its velocity (and subsequently its pressure and temperature) to change. If we want to figure how much Work and Compression a stage does, we need to understand how the air velocity (magnitude and direction) changes through a stage.

Two important reference frames in turbomachinery

- Relative - rotates with the rotor. $\vec{\omega}$ = relative velocity
- Absolute - fixed to frame of machine $\vec{c}$ = absolute velocity

Difference between these is the local blade velocity:

That is:

Velocity Triangle

$$\vec{c} = \vec{U} + \vec{\omega} \tag{12.14}$$

This leads to the velocity triangle of a single compressor stage (much like the Bermuda triangle, it is easy to get lost in the velocity triangle)

How do these two velocities change through a stage? (if we knew how the velocities changed, we could calculate how much work must have been put in to cause that change, and also the corresponding change in temperature and pressure of the air)

Consider the mean radius section (average fluid properties) of a stage. We draw the velocity triangles before, in the middle, and after a single stage.

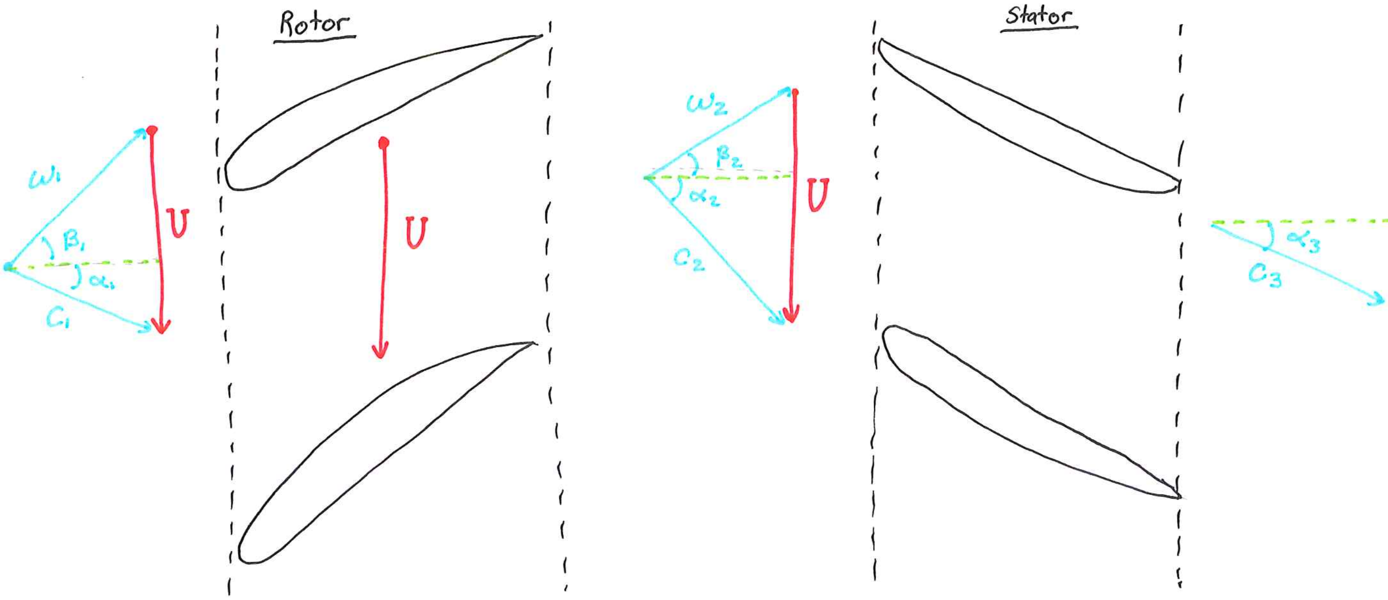


Figure: Rotor-Stater Pair

1. Flow enters the rotor $\vec{U}$ with absolute angle α_1 and absolute velocity $\vec{c}_1$
2. $\vec{\omega_1}$ can be found by vector subtraction of (12.14) $\vec{c_1} - \vec{U}$
3. The flow turns in the blade passage to the new relative velocity $\vec{\omega}_2$
4. Absolute velocity $\vec{c}_2$ found again by (12.14) $\vec{\omega_2} + \vec{U}$

Rotor puts in azimuthal velocity (KE), Stator removes it and increases pressure.

Combining velocity triangles at locations (1) and (2):

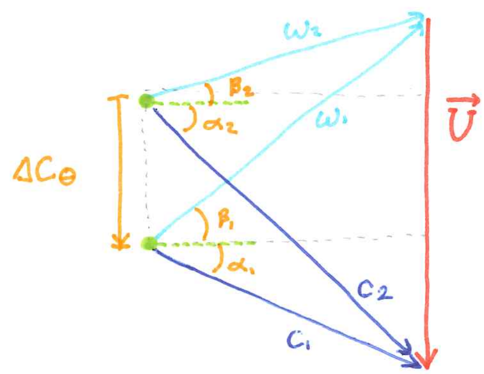


Figure: Velocity Triangle Comparisons

$$C_{z,1} = C_{z,2} = C_z \qquad \text{and} \qquad \Delta C_\theta = C_{\theta_2} - C_{\theta_1} \tag{12.15}$$

And thus we have the change in azimuthal velocity of the air through a single stage (which is related to the power, pressure, and temperature change, as follows)

Power required to drive the stage is:

$$P_s = \tau\,\Omega$$

Where

$$\tau = \text{Torque}$$

$$\Omega = \text{Rotational Speed}$$

$$= U/r$$

such that

Power to Drive Stage

$$P_s = \dot{m}\, r_m\, \Delta C_\theta\, \Omega = \dot{m}\, U\, \Delta C_\theta \tag{12.16}$$

Since the specific work required is:

$$\omega_s = U\, \Delta C_\theta \tag{12.17}$$

Since the Stator does not move, no work done in stator.

There is a torque on stator opposite to that on the fluid, but equal in magnitude.

This tells us the power for one single stage in the compressor.

1.2.2 Efficiency

We just figured out the Work done by each stage of a compressor, how about the efficiency, and subsequently the pressure rise across a single stage.

1st law of thermodynamics states:

$$P_s = \dot{m}\,(h_{o3} - h_{o1}) \tag{12.18}$$

Since no work or heat transfer in stator, then $T_{o3} = T_{o2}$

$$\dot{m}\,U\,\Delta C_\theta = \dot{m}\,c_P\,(T_{o2} - T_{o1}) \tag{12.19}$$

Then the temperature rise across a single stage is (12.20)

$$\frac{\Delta T_o}{T_{o1}} = \frac{U\,\Delta C_\theta}{c_P\,T_{o1}} \tag{12.20}$$

Efficiency defined the typical way (isentropic change in enthalpy divided by real change in enthalpy)

$$\eta_{\text{stage}} = \frac{h_{o3,s} - h_{o,1}}{h_{o3} - h_{o,1}} \tag{12.21}$$

Stage efficiency is then:

$$\eta_{\text{stage}} = \frac{P_{s,\text{isentropic}}}{P_{s,\text{actual}}}$$

Actual power required is then:

$$P_{s,\text{actual}} = \frac{\dot{m}\,U\,\Delta C_\theta}{\eta_{\text{stage}}} \tag{12.22}$$

(12.21) can also be written:

$$\frac{T_{o3,s}}{T_{o1}} = 1 + \eta_{\text{stage}}\,\frac{\Delta T_o}{T_{o1}} \tag{12.23}$$

With the isentropic relation, this becomes

$$\frac{P_{o3}}{P_{o1}} = \left[1 + \eta_{\text{stage}} \frac{\Delta T_o}{T_{o1}}\right]^{\frac{\gamma}{\gamma-1}} \tag{12.24}$$

Then, with (12.20) we have:

$$\frac{P_{o3}}{P_{o1}} = \left[1 + \eta_{\text{stage}} \frac{U\,\Delta c_\theta}{C_p\,T_{o1}}\right]^{\frac{\gamma}{\gamma-1}} \tag{12.25}$$

And this equation (12.25) shows that higher blade speed U and the higher change in azimuthal velocity yields a higher pressure ratio (as expected, run at higher RPMs, higher U means more energy input, get higher pressure ratio, similar for velocity change)

Clearly if the efficiency is = 1, this gives the max pressure change across the stage.

Three major factors determine

1. Boundary layer induced viscous dissipation losses in stagnation pressure (friction between air and surfaces of rotors, stators, hub, housing)
2. Massive flow irreversibilities from blades, flow separation (spin too fast, flow separates)
3. Compressibility effects (shocks, but generally avoid inside engine, but still, the relative blade speed can be high 300 m/s, which would be M 0.8, M 0.4 to 0.8 possible, no shocks, but definitely compressible!)

1.2.3 Compressor Starting

Remember the geometry from the very first figure. Area decreases with axial distance to give the same velocity at each stage

So, the velocity triangle at each stage changes since c_z is NOT constant during startup. The velocity triangle at the First Stage vs. Last Stage:

- Because of varying axial velocity it is possible for only one stage near the middle to operate ideally for all speeds (called the "pivot stage")
- At lower speeds, blades forward of pivot tend toward stall (blade speed is too high), while blades aft of pivot tend toward choking (axial c_z is too high)
- At speeds higher than design, trends are opposite.

Solutions:

1. Bleed out some air to keep c_z constant (decrease mass flow)
2. Multispool compressor, run each spool at its own suitable speed

1.2.4 Compressor Map

The pumping characteristics of axial-flow compressors, indeed all compressors, best represented by an experimental data plot called a compressor map. Data obtained as follows: Compressor driven by electric motor, controls RPM ? or U Mass flow rate controlled by valve Inlet and exit properties measured along with ? and power We are interested, for a given compressor, how the pressure ratio Po3/Po1 changes as mass flow and U change.

$$a \tag{12.26}$$

1.3 Combustors

Goals of designing Combustor:

1. Burn all fuel injected in combustor
2. Achieve proper temperature of working fluid entering turbine
3. Proper temperature profile
4. Minimized pressure drop
5. Minimize size
6. Maintain flame over wide range of conditions

Figure of Merit:

Burner Efficiency:

$$\eta_b = \frac{\Delta H}{\Delta Q} = \frac{\text{Actual Energy Released}}{\text{Available Energy in Fuel}} \tag{12.27}$$

Overall Pressure Ratio:

$$\pi_b = \frac{P_{o_{\text{exit}}}}{P_{o_{\text{entrance}}}} \tag{12.28}$$

Typical values of f (fuel-air ratio) for engines are 1/50 (0.02, 0.025, 2%).

This is much less than stoichiometric 0.067 for hydrocarbon fuels.

Thus, fuel-lean:

$$\phi = \frac{f}{f_{\text{stoich}}} = \frac{0.025}{0.067} = 0.37 < 1$$

It is difficult to combust such a lean mixture of hydrocarbon-air.

Example of inflammability limits of gasoline-air mixture:

(inflammable = flammable, both mean easily ignited, you should use nonflammable to describe something that is not ignited easily)

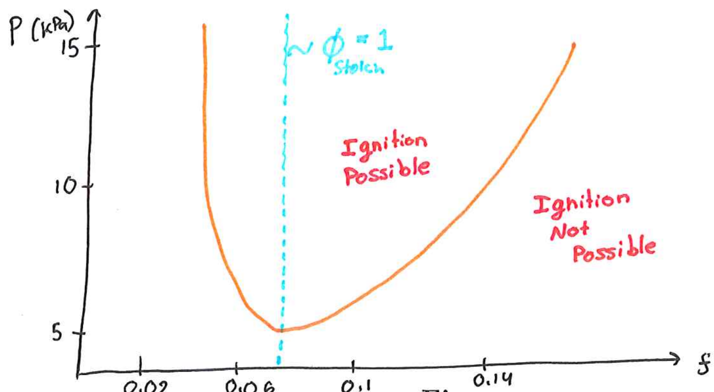


Figure:

If f were required to be 0.025 and all air mixed with all fuel, mixture would NOT combust!

So, split air into two parts:

1. Primary air – undergoes combustion
2. Secondary air – dilute and cool combusted primary air, give proper temperature and temperature profile

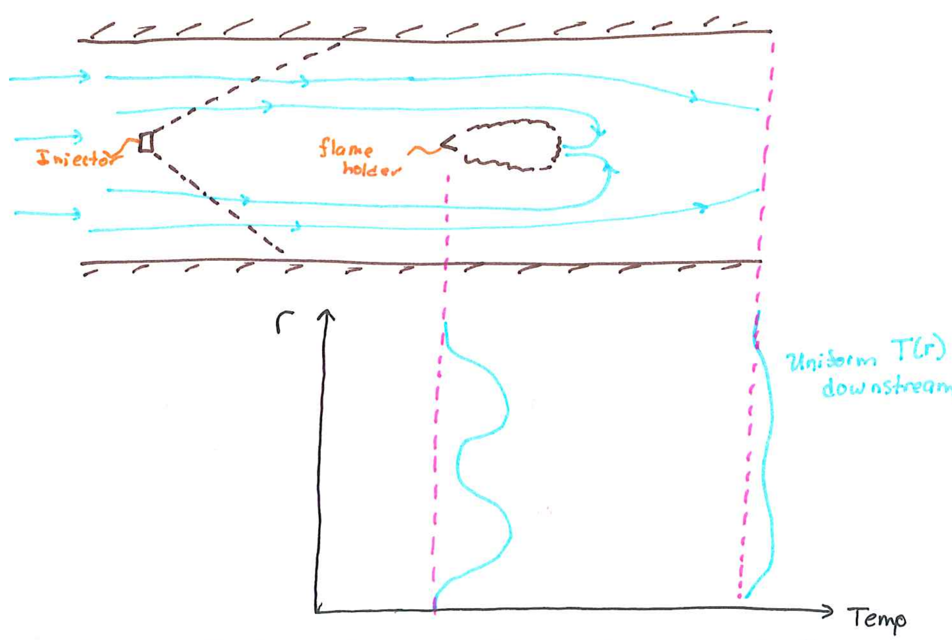


Figure: Combustor Diagram and Radial Temperature Profile

More sophisticated systems have separate channels for primary and secondary air (similar to a bypass). The secondary air is slowly fed into the primary stream downstream of combustion volume.

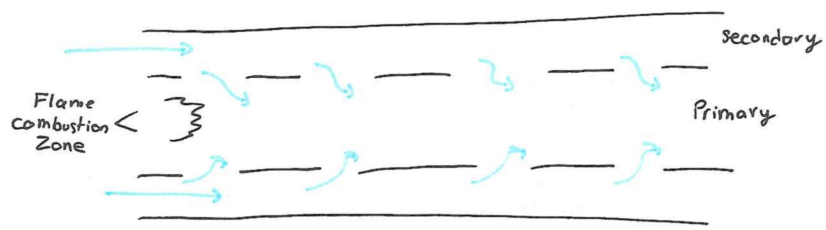


Figure: Sophisticated Combustor Geometry

1.3.1 Flame propagation

A higher velocity of working fluid thru combustor is desirable for increased performance, smaller engine size per unit thrust.

This velocity is limited by the flame speed.

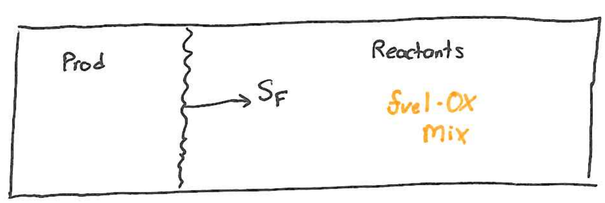
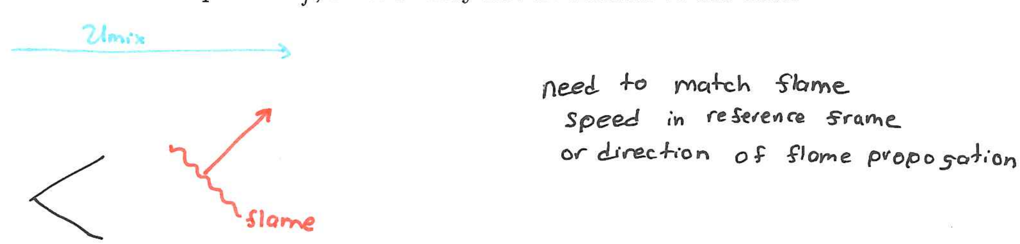


Figure: Flame Propagation

If mixture velocity is too high, flame "blown out" exit.

If mixture velocity is too low, flame travels upstream and is extinguished.

However, u_{mix} does not have to be equal to s_f, since sf may not be normal to the flow.



Finally, combustor analysis to determine total pressure loss proceeds with Rayleigh and Fanno flow analyses, see IV E. F. G.

1.4 Axial Turbine

Similar to compressor

Issues:

- Very high loads
- High temperature
- Need to actively cool
- Vibrations induced by compressibility effects

Rows of stationary blades (nozzle or stator) followed by rotor

Again a stage is one pair of these (nozzle + rotor).

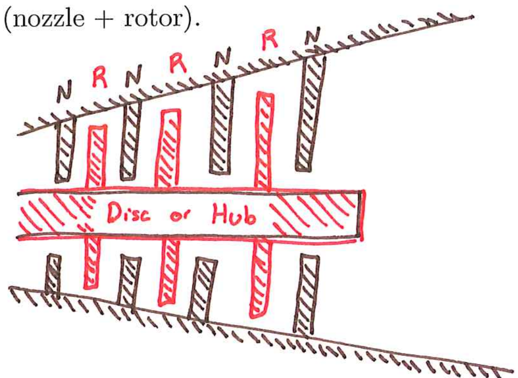


Figure: Turbine Diagram

Again, we consider a single stage:

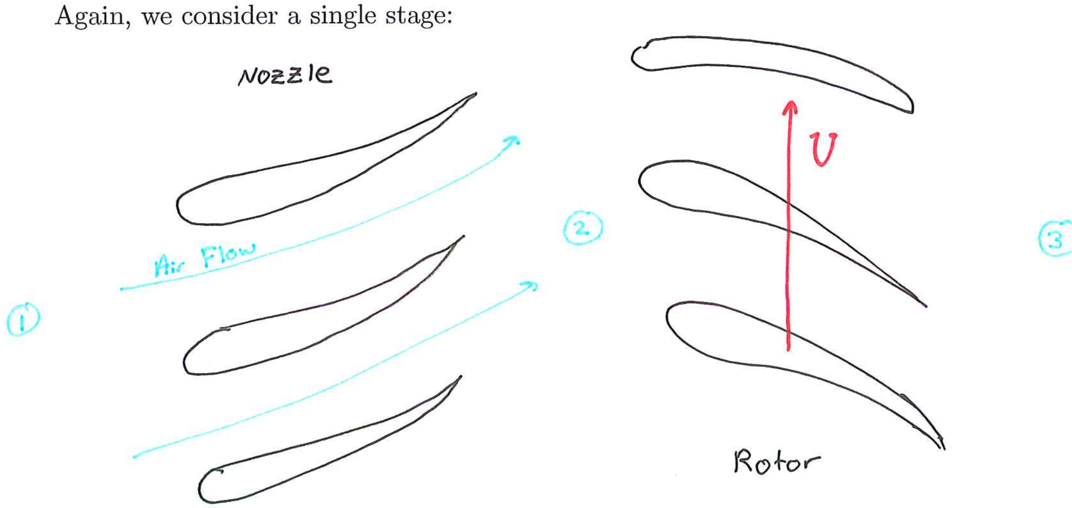


Figure: Turbine Single Stage

Velocity triangle is then:

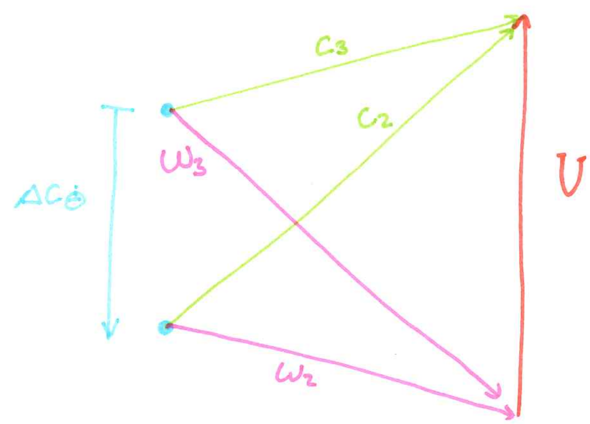


Figure: Velocity Triangle

Lots of upward azimuthal velocity @ (2), decrease through rotor to (3). Then change in azimuthal velocity is:

$$\Delta C_\theta = C_{\theta_3} - C_{\theta_2}$$

Note that now, U and ΔC_θ now in opposite directions, the change in tangential momentum of fluid causes a torque on the rotor.

Power output from this stage is: (12.16)

$$P = \dot{m}\, U\, \Delta C_\theta \tag{12.16}$$

Work output from stage is (12.17)

$$W = U \Delta C_\theta = C_P\, (T_{o3} - T_{o2}) \tag{12.17}$$

Then (12.20).

$$\frac{\Delta T_o}{T_{o1}} = \frac{U\, \Delta C_\theta}{C_p\, T_{o1}} \tag{12.20}$$

$$\Delta T_o = T_{o3} - T_{o2} \qquad T_{o2} = T_{o1}$$

Efficiency however can be defined two ways for a turbine:

If the gas still has some kinetic energy after the turbine, then in one case we consider it a loss but in the other efficiency case, it is not a loss.

1st Case: Exhaust KE out of turbine is a loss (applicable for a power plant, generator, propeller)

So we use the static temperature T3 for isentropic process

$$W_{\text{ideal}} = C_P\left(T_{o1} - T_{3,s}\right)$$

Total - to -Static Efficiency (TS)

$$\eta_{ts} = \frac{T_{o1} - T_{o3}}{T_{o1} - T_{3s}}$$

Written in terms of pressure

$$\eta_{ts} = \frac{1 - T_{o3}/T_{o1}}{1 - \left(P_3/P_{o1}\right)^{\frac{\gamma-1}{\gamma}}}$$

2nd Case: Exhaust KE NOT considered a loss, applicable to turbojets, turbofans, gases intended to be at high KE

$$W_{\text{ideal}} = C_P\left(T_{o1} - T_{o3,s}\right)$$

Total-to-Total efficiency

$$\eta_{tt} = \frac{T_{o1} - T_{o3}}{T_{o1} - T_{o3s}} = \frac{1 - T_{o3}/T_{o1}}{1 - \left(P_{o3}/P_{o1}\right)^{\frac{\gamma-1}{\gamma}}}$$

Comparison of two efficiencies:

$$\eta_{tt} > \eta_{ts}$$

This is because η_{ts} has KE defined as a loss.